Due Date: 20 November 2021 Saturday
Time: 10:00-12:30
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 202 Complex Analysis – Midterm Exam – Solution Key

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are **5** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

"At every district there will be a geometer, and we will give away triangles to all."

Cartoon by Selçuk Erdem

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Q-1) Calculate the principal value of $(1+i)^i$ and write the result in the rectangular form $a+ib$ where $a, b \in \mathbb{R}$. *(Recall that "principal value" means that the argument of a complex number is to be considered between $-\pi$ and π.)*

Solution:

Note that $i = e^{i\pi/4} = \dfrac{(1+i)}{\sqrt{2}}$, so $(1+i) = \sqrt{2}e^{i\pi/4}$.

Now $\log(1+i)^i = i\log(1+i) = i\log(\sqrt{2}e^{i\pi/4}) = i(\ln\sqrt{2} + i\pi/4) = i\ln\sqrt{2} - \pi/4$.

Finally $(1+i)^i = \exp(\log[(1+i)^i]\ ) = \exp(i\ln\sqrt{2} - \pi/4) = \exp(i\ln\sqrt{2})\ \exp(-\pi/4)$.

Hence $(1+i)^i = e^{-\pi/4}\cos\ln\sqrt{2} + ie^{-\pi/4}\sin\ln\sqrt{2} \approx 0.42 + 0.15i$.

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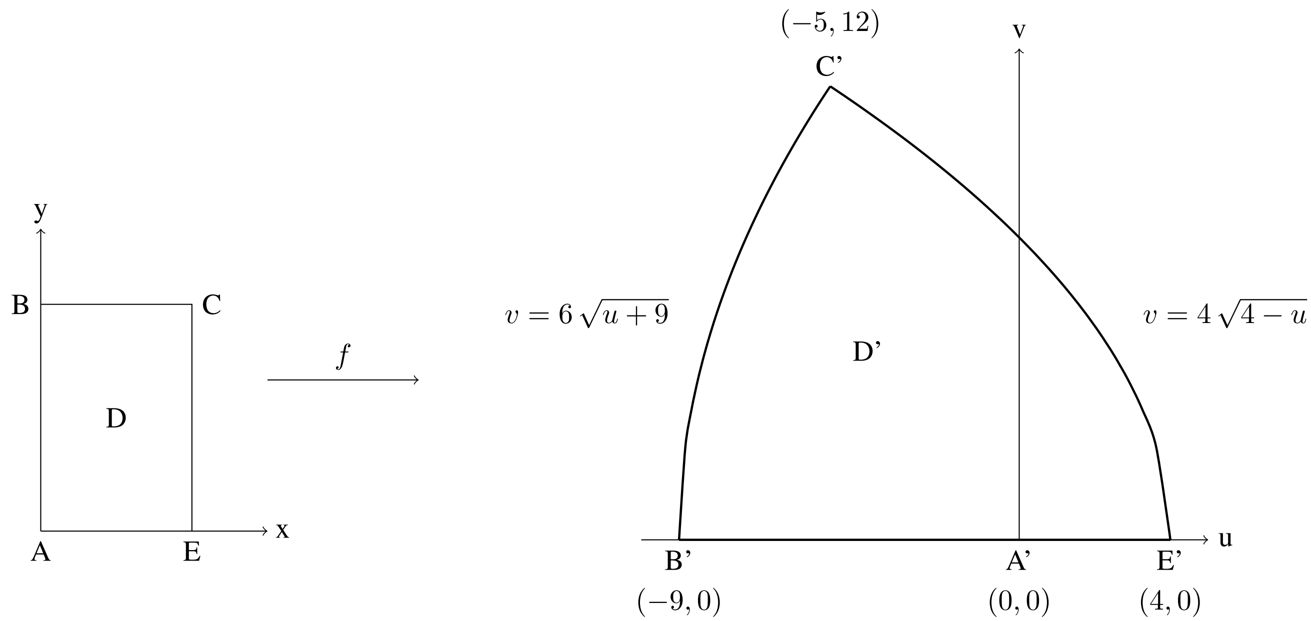
Q-2) Let $D = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 2,\ 0 \leq \operatorname{Im} z \leq 3\ \}$ and $f(z) = z^2$. Describe $f(D)$.

Solution:



Holomorphic functions preserve orienttion so the inside of D will be mapped to the inside of the image of the boundary of D. Therefore we map the boundary of D by z^2.

Note that $f(z) = z^2 = (x^2 - y^2) + i(2xy)$, so

$$u = x^2 - y^2, \ \ v = 2xy.$$

AB : Here $x = 0$, $0 \leq y \leq 3$.
$u = -y^2$, $v = 0$. Hence $-9 \leq u \leq 0$.
AB maps to A'B'.

BC: Here $y = 3$, $0 \leq x \leq 2$.
$u = x^2 - 9$, $v = 6x$, so $v = 6\sqrt{u+9}$ and $-9 \leq u \leq -5$.
BC maps to B'C'.

CE: Here $x = 2$, $0 \leq y \leq 3$.
$u = 4 - y^2$, $v = 4y$, so $v = 4\sqrt{4-u}$ and $-5 \leq u \leq 4$.
CE maps to C'E'.

EA: Here $y = 0$, $0 \leq x \leq 2$.
$u = x^2$, $v = 0$ and $0 \leq u \leq 4$.
EA maps to E'A'.

Coordinates of these significant points are as follows.

$$A' = (0, 0), \quad B' = (-9, 0), \quad C' = (-5, 12), \quad E' = (4, 0)$$

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Q-3) Evaluate

$$\frac{1}{2\pi i}\int_{|z|=2}\frac{z^{1/3}}{z^2+1}\,dz,$$

where the cube root is taken using the principal value.

Solution:

First observe that $i=\exp(i\pi/2)$, hence $i^{1/3}=\exp(i\pi/6)=\frac{\sqrt{3}}{2}+i\frac{1}{2}$.

And also $-i=\exp(-i\pi/2)$ and $(-i)^{1/3}=\exp(-i\pi/6)=\frac{\sqrt{3}}{2}-i\frac{1}{2}$.

Now define $g(z)=\frac{z^{1/3}}{z+i}$ and $h(z)=\frac{z^{1/3}}{z-i}$. Observe that we can write

$$\frac{z^{1/3}}{z^2+1}=\frac{g(z)}{z-i}=\frac{h(z)}{z+i}.$$

Hence

$$\mathrm{Res}(\frac{z^{1/3}}{z^2+1},z=i)=g(i)=\frac{1}{4}-i\frac{\sqrt{3}}{4},$$

and

$$\mathrm{Res}(\frac{z^{1/3}}{z^2+1},z=-i)=h(-i)=\frac{1}{4}+i\frac{\sqrt{3}}{4}.$$

Finally

$$\frac{1}{2\pi i}\int_{|z|=2}\frac{z^{1/3}}{z^2+1}\,dz=g(i)+h(-i)=\frac{1}{2}.$$

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Q-4) For any positive integer $n > 1$ define the function

$$f(n) = \frac{1}{2\pi i} \int_{|z|=n} \frac{z^{2n-1}}{(z^2-1)(z^2-2)\cdots(z^2-n)}\, dz.$$

Evaluate $f(n)$, $n = 2, 3, \ldots$.

Solution:

Let $\phi_n(z) = \dfrac{z^{2n-1}}{(z^2-1)(z^2-2)\cdots(z^2-n)}$.

Then $\phi_n(\frac{1}{w})\,\frac{1}{w^2} = \frac{1}{w}\,\alpha_n(w)$, where $\alpha_n(w) = \dfrac{1}{(1-w^2)(1-2w^2)\cdots(1-nw^2)}$.

Notice that $\alpha_n(w)$ is analytic around $w = 0$ and $\alpha(0) = 1$.

Now since all the poles of $f(n)$ are inside the given disk, we can write, after substituting $z = 1/w$

$$f(n) = \frac{1}{2\pi i} \int_{|w|=\frac{1}{n}} \phi_n(\frac{1}{w})\frac{1}{w^2}\, dw = \frac{1}{2\pi i} \int_{|w|=\frac{1}{n}} \frac{\alpha_n(w)}{w}\, dw = \alpha_n(0) = 1.$$

For fun you may consider the following.

Let $F(z, n) = \dfrac{z^{2n-1}}{(z^2-1)(z^2-2)\cdots(z^2-n)}$, $n > 1$. This function has simple poles at $\pm\sqrt{m}$ for $m = 1, \ldots, n$. By direct calculation we find that

$$\mathrm{Res}(F(z,n), z = \pm\sqrt{m}) = \frac{1}{2}\,\frac{m^{n-1}}{\prod\limits_{\substack{k=1\\k\neq m}}^{n}(m-k)}.$$

Since the above $f(n)$ is equal to the sum of all its residues, we find that

$$f(n) = \sum_{m=1}^{n} \frac{m^{n-1}}{\prod\limits_{\substack{k=1\\k\neq m}}^{n}(m-k)}.$$

Now the above calculation using the residue at infinity shows that this incredible sum is always equal to 1.

Try it!

For example when $n = 5$, we have

$$f(5) = \frac{1}{24} + \frac{-8}{3} + \frac{81}{4} + \frac{-128}{3} + \frac{625}{24} = 1.$$

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Q-5) Let f be a meromorphic function in a simply connected region D with no zeros and finitely many poles. Let C be a simple closed contour in D enclosing all poles of f. Calculate

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz.$$

Solution:

Without loss of generality we can assume that f has only one pole, say at z_0 of order m. We can write

$$\begin{aligned} f(z) &= \frac{b_m}{(z-z_0)^m}+\cdots+\frac{b_1}{z-z_0}+a_0+a_1(z-z_0)+\cdots,\ b_m\neq 0 \\ &= \frac{1}{(z-z_0)^m}\left(b_m+b_{m-1}(z-z_0)+\cdots\right) \\ &= \frac{1}{(z-z_0)^m}\phi(z), \end{aligned}$$

where we observe that $\phi(z)$ is analytic around z_0 and $\phi(z_0)=b_m\neq 0$.

Then

$$f'(z)=\frac{-m}{(z-z_0)^{m+1}}\phi(z)+\frac{1}{(z-z_0)^m}\phi'(z),$$

and

$$\frac{f'(z)}{f(z)}=\frac{-m}{z-z_0}+\frac{\phi'(z)}{\phi(z)}.$$

Since ϕ'/ϕ is analytic inside C, its integral is zero. We get

$$\frac{1}{2\pi i}\int_C\frac{f'(z)}{f(z)}\,dz=\frac{1}{2\pi i}\int_C\left[\frac{-m}{z-z_0}+\frac{\phi'(z)}{\phi(z)}\right]\,dz=\frac{1}{2\pi i}\int_C\frac{-m}{z-z_0}\,dz=-m.$$

If we have more than one pole, we repeat the above calculation around each pole and as a result of this the value of the integral in the question becomes the negative sum of the orders of all poles inside C.