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Math 206 Complex Calculus – Midterm Exam II
Solutions

Q-1) Find the residue of $f(z) = \dfrac{1}{z^2 \sinh z}$ at $z = 0$.

Solution: This is Example 2 on page 171 of your textbook. Here is a slightly easier solution:

$$
\begin{aligned}
\sinh z &= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \\
&= z(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots) \\
z^2 \sinh z &= z^3(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots) \\
\frac{1}{z^2 \sinh z} &= \frac{1}{z^3(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots)} \\
&= \frac{1}{z^3}(1 + a_1 z + a_2 z^2 + \cdots) \\
&= \frac{1}{z^3} + \frac{a_1}{z^2} + \frac{a_2}{z} + \cdots.
\end{aligned}
$$

We are interested in finding a_2. For this observe that

$$
\begin{aligned}
1 &= (1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots)(1 + a_1 z + a_2 z^2 + \cdots) \\
&= 1 + a_1 z + (\frac{1}{3!} + a_2)z^2 + \cdots.
\end{aligned}
$$

Hence $a_1 = 0$ and $a_2 = -\dfrac{1}{6}$. Thus

$$
\operatorname*{Res}_{z=0} \frac{1}{z^2 \sinh z} = -\frac{1}{6}.
$$

Q-2) Let $f(z)$ be the function

$$f(z) = \frac{z^{99}}{(z-1)(z-2)\cdots(z-100)},$$

and let C be the positively oriented circle $|z| = 528$. Evaluate the integral

$$\int_C f(z)\,dz.$$

Solution: We first notice that all the 100 poles are inside the contour $|z| = 528$. Clearly there is no hope of calculating the residues at 100 different points and adding them up correctly. Instead we use Theorem 2 on page 185 of your book, which says

$$\int_C f(z)dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right].$$

A direct computation shows that

$$f\left(\frac{1}{z}\right) = \frac{z}{(1-z)(1-2z)\cdots(1-100z)}$$

$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{z}\frac{1}{(1-z)(1-2z)\cdots(1-100z)}.$$

Now it is immediate that the residue at 0 is 1, see for example the theorem on page 190 of your book. Thus

$$\sum_{n=1}^{100}\operatorname*{Res}_{z=n}f(z) = \operatorname*{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right] = 1.$$

And finally we get

$$\int_C f(z)\,dz = 2\pi i.$$

Q-3) Evaluate the improper integral

$$\int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx.$$

Solution: For this integral we use the closed path that is given in Figure 70 on page 224 of your book. This problem is slightly more difficult than the solved example on that page and considerably easier than exercise 2 on page 226.

Using the notation and terminology of that example, we first write equation (4) on page 225: Setting

$$f(x) = \frac{x^{1/3}}{x^2+1}\,dx$$

we have

$$\int_\rho^R \frac{x^{1/3}}{x^2+1}\,dx + \int_{C_R} f(z)\,dz - \int_\rho^R \frac{x^{1/3}e^{i2\pi/3}}{x^2+1}\,dx + \int_{C_\rho} f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=i}f(z) + \operatorname*{Res}_{z=-i}f(z)\right).$$

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Noting that both poles are simple we first calculate the residues:

$$\begin{aligned}
\operatorname*{Res}_{z=i} f(z) &= \left.\frac{z^{1/3}}{2z}\right|_{z=i} \\
&= \frac{(e^{i\pi/2})^{1/3}}{2i} \\
&= \frac{e^{i\pi/6}}{2i} = \frac{1}{2i}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) \\
&= \frac{1}{4} + \frac{\sqrt{3}}{4i}.
\end{aligned}$$

$$\begin{aligned}
\operatorname*{Res}_{z=-i} f(z) &= \left.\frac{z^{1/3}}{2z}\right|_{z=-i} \\
&= \frac{(e^{i3\pi/2})^{1/3}}{-2i} \\
&= \frac{e^{i\pi/2}}{-2i} = -\frac{1}{2i}\, i \\
&= -\frac{1}{2}.
\end{aligned}$$

And

$$2\pi i\left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z)\right) = \frac{\pi\sqrt{3}}{2} - i\frac{\pi}{2}.$$

We then analyze the integrals on the semicircles:

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{R^{1/3}}{R^2-1} 2\pi R = \frac{R^{4/3}}{R^2-1} 2\pi,$$

$$\left|\int_{C_\rho} f(z)\,dz\right| \leq \frac{\rho^{1/3}}{1-\rho^2} 2\pi\rho = \frac{\rho^{4/3}}{1-\rho^2} 2\pi.$$

Both of these integrals converge to zero as $\rho \to 0$ and $R \to \infty$. Moreover note that $e^{i2\pi/3} = -(1/2) + i(\sqrt{3}/2)$. Setting $I = \int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx$, and putting all our findings into the first equation we had gives

$$I - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) I = \frac{\pi\sqrt{3}}{2} - i\frac{\pi}{2}.$$

Or equivalently

$$\frac{3}{2}I - i\frac{\sqrt{3}}{2}I = \frac{\pi\sqrt{3}}{2} - i\frac{\pi}{2}.$$

Which then finally gives

$$I = \frac{\sqrt{3}}{3}\pi.$$

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Q-4) Solve the following initial value problem using Laplace transform techniques:

$$f''(t) + 2f'(t) + f(t) = 1 + t, \quad f(0) = 0, \ f'(0) = 1.$$

Solution:

$$
\begin{aligned}
\mathcal{L}(f(t)) &= F(s) \\
\mathcal{L}(f'(t)) &= sF(s) - f(0) = sF(s) \\
\mathcal{L}(f''(t)) &= s^2 F(s) - sf(0) - f'(0) = s^2 F(s) - 1 \\
\mathcal{L}(t) &= \frac{1}{s^2} \text{ and} \\
\mathcal{L}(1) &= \frac{1}{s}.
\end{aligned}
$$

Taking the Laplace transform of both sides using the above identities and solving for $F(s)$ gives

$$
\begin{aligned}
(s^2 F(s) - 1) + 2(sF(s)) + F(s) &= \frac{1}{s} + \frac{1}{s^2} \\
F(s)(s^2 + 2s + 1) - 1 &= \frac{s+1}{s^2} \\
F(s)(s+1)^2 - 1 &= \frac{s+1}{s^2} \\
F(s) &= \frac{1}{s^2(s+1)} + \frac{1}{(s+1)^2} \\
&= \frac{1}{s^2} - \frac{1}{s} + \frac{1}{(s+1)} + \frac{1}{(s+1)^2}.
\end{aligned}
$$

Taking the inverse Laplace transform of this gives

$$f(t) = t - 1 + e^{-t} + te^{-t}$$

which is the solution to the given initial value problem as can easily be verified.

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