Math 206 Complex Calculus
Quiz-4
Solutions

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1) Use residues to evaluate the improper integral $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$.

Solution: Letting $C_R = \{z \in \mathbb{C} |\ |z| = R,\ R > 1\}$, we have

$$\begin{aligned}
\int_0^\infty \frac{dx}{x^2+1} &= \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{x^2+1} \\
&= \frac{1}{2}\left(2\pi i \mathrm{Res}_{z=i}\frac{1}{z^2+1} - \lim_{R\to\infty}\int_{C_R}\frac{dz}{z^2+1}\right) \\
&= \pi i \left(\left.\frac{z-i}{z^2+1}\right|_{z=i}\right) \\
&= \pi i \left(\left.\frac{1}{z+i}\right|_{z=i}\right) \\
&= \pi i \left(\frac{1}{2i}\right) \\
&= \frac{\pi}{2}.
\end{aligned}$$

2) Use residues to evaluate the improper integral $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2}$.

Solution: Letting $C_R = \{z \in \mathbb{C} |\ |z| = R,\ R > 1\}$, we have

$$\begin{aligned}
\int_0^\infty \frac{dx}{(x^2+1)^2} &= \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2} \\
&= \frac{1}{2}\left(2\pi i \mathrm{Res}_{z=i}\frac{1}{(z^2+1)^2} - \lim_{R\to\infty}\int_{C_R}\frac{dz}{(z^2+1)^2}\right) \\
&= \pi i \left(\left.\frac{d}{dz}\right|_{z=i}\frac{(z-i)^2}{(z^2+1)^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \pi i \left(\frac{d}{dz}\bigg|_{z=i} \frac{1}{(z+i)^2} \right) \\
&= \pi i \left(\frac{d}{dz}\bigg|_{z=i} \frac{-2(z+i)}{(z+i)^4} \right) \\
&= \pi i \left(\frac{1}{4i} \right) \\
&= \frac{\pi}{4}.
\end{aligned}
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Remark: These are Exercises 1 and 2 on page 208 of your textbook.