Math 206 Complex Calculus
Quiz-5
Solutions

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1) Consider the region $D = \{ \rho e^{i\theta} \mid 0 < \rho < 1,\ 0 \le \theta \le \pi/2 \}$. Describe the image of D under the transformation $f(z) = z + 1/z$.

Solution: Note that in general the image is described as

$$f(\rho e^{i\theta}) = \left(\rho + \frac{1}{\rho}\right)\cos\theta + i\left(\rho - \frac{1}{\rho}\right)\sin\theta = u + iv.$$

We first map the boundary of D:

When $0 < \rho \le 1$ and $\theta = 0$, then $u = (\rho + 1/\rho)$ and $v = 0$. Thus the image is $u \ge 2,\ v = 0$, and as ρ increases from 0 to 1, u decreases from ∞ to 2.

When $\rho = 1$, then $v = 0$ and $u = 2\cos\theta$. As θ increases from 0 to $\pi/2$, u decreases from 2 to 0.

When $\theta = \pi/2$, then $u = 0$ and as ρ decreases from 1 to 0, v decreases from 0 to $-\infty$.

We see that the boundary of D is mapped onto positive x-axis and the negative y-axis.

Let $R_\theta = \{ \rho e^{i\theta} \mid 0 < \rho < 1 \}$ be any ray inside D with $0 < \theta < \pi/2$. We see then that $u > 0$ and $v < 0$, and the image is an arm of a hyperbola with equation

$$\frac{u^2}{(2\cos\theta)^2} + \frac{v^2}{(2\sin\theta)^2} = 1$$

coming from $-\infty$ and touching the x axis at $u = (\rho + 1/\rho)\cos\theta$ with $0 < u < 2$.

Conclusion: Every point (u, v) with $u \ge 0$ and $v \le 0$ is in the image $f(D)$.

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2) Find a one-to-one analytic transformation which maps the region $D = \{ z \in \mathbb{C} \mid 0 < |z| \leq 1,\ \operatorname{Im} z \geq 0 \}$ onto the lower half plane $H^- = \{ w \in \mathbb{C} \mid \operatorname{Im} w \leq 0 \}$ in such a manner that the boundary of D maps onto the boundary of H^-.

Solution: Consider the transformation $f(z) = z + 1/z$. Then for a general point $z = \rho e^{i\theta} \in D$,

$$f(\rho e^{i\theta}) = \left(\rho + \frac{1}{\rho}\right)\cos\theta + i\left(\rho - \frac{1}{\rho}\right)\sin\theta = u + iv.$$

We first map the boundary of D:

The semicircle is mapped onto $[-2, 2]$.

$[-1, 0)$ is mapped onto $(-\infty, -2]$.

$(0, 1]$ is mapped onto $[2, \infty)$.

We see that the boundary of D is mapped onto the real axis in the w-plane. Since f is analytic, the inside of D is then mapped totally either to the lower half plane or the upper half plane in the w-plane.

Let $\rho e^{i\theta}$ be any point interior to D. Then $0 < \rho < 1$ and $0 < \theta < \pi$ and we see that this forces $v < 0$. Thus the inside of D maps to $\operatorname{Im} w < 0$.

Conclusion: The required transformation is $f(z) = z + \dfrac{1}{z}$.

Remark: Last week I gave the above transformation as an optional exercise for you to work out. This transformation is also mentioned in Exercise 19 on page 251 and in the table of transformations in Figure 16 on page 374.