MATH 206, HW#1
SOLUTIONS

p.17: 6. If $Re\, z_i > 0$, then z_i is in the first or fourth open quadrants in the complex plane; thus, $-\pi/2 < Arg\, z_i < \pi/2$ for $i = 1, 2$. It follows that

$$-\pi < Arg\, z_1 + Arg\, z_2 < \pi$$

which implies that $Arg\, z_1 + Arg\, z_2$ is the principal argument of $z_1 z_2$. That is $Arg\,(z_1 z_2) = Arg\, z_1 + Arg\, z_2$.

p.22: 2. a. In general polar form $-1 = exp(i\pi + i2k\pi)$, $k = 0, 1, 2, ...$ so that the third roots $(-1)^{1/3}$ are $c_k = exp(i\frac{\pi}{3} + i\frac{2k\pi}{3})$, $k = 0, 1, 2$. Thus,

$$c_0 = exp(i\frac{\pi}{3}) = \frac{1}{2} + i\frac{\sqrt{3}}{2},\ c_1 = exp(i\pi) = -1,\ c_2 = exp(i\frac{5\pi}{3}) = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

d. We have $-8 - i8\sqrt{3} = 16exp(i\frac{4\pi}{3} + i2k\pi)$, $k = 0, 1, 2, ...$ so that the fourth roots $(-8 - i8\sqrt{3})^{1/4}$ are $c_k = 2exp(i\frac{\pi}{3} + i\frac{k\pi}{2})$, $k = 0, 1, 2, 3$. Thus,

$$c_0 = exp(i\frac{\pi}{3}) = 1 + i\sqrt{3},\ c_1 = exp(i\frac{5\pi}{6}) = -\sqrt{3} + i,$$

$$c_2 = exp(i\frac{4\pi}{3}) = -1 - i\sqrt{3},\ c_3 = exp(i\frac{11\pi}{6}) = \sqrt{3} - i.$$

p.23: 6. We have $-4 = 4exp(i\pi + i2k\pi)$, $k = 0, 1, 2, ...$, which gives

$$c_0 = exp(i\frac{\pi}{4}) = 1 + i,\ c_1 = exp(i\frac{3\pi}{4}) = -1 + i,$$

$$c_2 = exp(i\frac{5\pi}{4}) = -1 - i,\ , c_3 = exp(i\frac{7\pi}{4}) = 1 - i.$$

Writing $z^4 + 4 = [(z - c_0)(z - c_3)][(z - c_1)(z - c_2)]$ and performing the multiplications in brackets, we have

$$z^4 + 4 = [(z - 1)^2 + 1][(z + 1)^2 + 1] = (z^2 - 2z + 2)(z^2 + 2z + 2).$$

p.25: 4. (a) The set is the whole complex plane except the real numbers x with $x \leq 0$. Its closure is the whole complex plane. However sometimes it is important to identify the complex plane with the Riemann sphere minus the north pole. With this identification the closure of any unbounded set, such as the one in this problem, must include the north pole which is infinity.

(b) The set is the set of points $x + iy$ satisfying $|x| < \sqrt{x^2 + y^2}$. Since both sides are positive we can square both sides preserving the inequality to get $x^2 < x^2 + y^2$, which gives $y \neq 0$. So this set is the whole plane except the x-axis. Its closure is the whole plane. See the note above about infinity.

(c) The set is the set of points $x+iy$ satisfying $\frac{x}{x^2+y^2} \leq \frac{1}{2}$, or $(x-1)^2+y^2 \geq 1$. This is the set of points on and outside the circle of radius 1 centered at $(x,y) = (1,0)$. Its closure it itself, plus infinity if the Riemann sphere is considered.

(d) The set is the set of points $x+iy$ satisfying $(x-y)(x+y) > 0$ which is the set $S := \{x+iy : x > y \ \& \ x > -y\} \cup \{x+iy : x < y \ \& \ x < -y\}$. Note that this is that region between (and excluding) the lines of slope 1 and -1 containing the positive and negative numbers. Its closure is itself together with the lines $y = \pm x$, plus infinity if the Riemann sphere is considered.

p.25: 7. c. The region is the first quadrant without the imaginary axis. The accumulation points are all points in the first quadrant including both axes and the origin. Since every interior point is an accumulation point, we need only to prove that the origin and the points $x+i0$ and $0+iy$ with $x > 0$, $y > 0$ are accumulation points. Given $\epsilon > 0$, let $0 < \epsilon_0 < \epsilon$ (e.g., $\epsilon_0 = \epsilon/2$ would do). Then, the deleted epsilon neighborhood of $x+i0$ and $0+iy$ include $x+i\epsilon_0$ and ϵ_0+iy, respectively, both of which are in the set $0 \leq arg\, z \leq \pi/2$. Also, the deleted ϵ neighborhood of the origin includes $\frac{\epsilon_0}{\sqrt{2}}(1+i)$ which is also in the set.

d. The accumulation points determined by plotting the points are $1+i$ and $-1-i$. Given $\epsilon > 0$, let N be an even integer such that $N > \frac{\sqrt{2}}{\epsilon}$. Then,

$$0 < |(-1)^N(1+i)\frac{N-1}{N} - (1+i)| = \frac{\sqrt{2}}{N} < \epsilon$$

so that $z_e := (-1)^N(1+i)\frac{N-1}{N}$ is both in the deleted ϵ neigborhood of $1+i$ and is a member of the given set. Similarly, to show that $-1-i$ is an accumulation point, let N be an odd integer such that $N > \frac{\sqrt{2}}{\epsilon}$. Then, $z_o := (-1)^N(1+i)\frac{N-1}{N}$ is both in the deleted ϵ neigborhood of $1+i$ and is a member of the given set by the same inequality displayed above.