Date: 26 March 2005, Saturday

Math 206 Complex Calculus – Midterm Exam I – Solutions

Q-1) Solve the following equations and write your answers in rectangular, $z = x + iy$, form:

a) $z^4 = -8 + i8\sqrt{3}$.

b) $\cosh z = \frac{3}{4}i$.

Answers: (a) $-8 + i8\sqrt{3} = 2^4 exp(i[\frac{2\pi}{3} + 2n\pi])$, $n \in \mathbb{Z}$.

Then $z = 2exp(i[\frac{2\pi}{3} + 2n\pi]/4)$ for $n = 0, 1, 2, 3$.

This gives $z = \pm(\sqrt{3} + i)$ and $\pm(1 - \sqrt{3}i)$.

(b) $\cosh z = \cosh x \cos y + i \sinh x \sin y$.

We start with $\cosh x \cos y = 0$. Since $\cosh x \geq 1$, we must have $\cos y = 0$ or $y = (2n + 1)\pi/2$, for $n \in \mathbb{Z}$. Note that for this value of y, $\sin y = (-1)^n$.

Now solving separately for $\sinh x = 3/4$ and $\sinh x = -3/4$, we get
$z = (-1)^n \ln 2 + i((2n + 1)\pi/2)$, $n \in \mathbb{Z}$.

Q-2) Find the value of the integral $\displaystyle\int_{|z|=1} \frac{(\cos z)\,(e^z)}{z^4}\,dz$.

Answer: Let $f(z) = (\cos z)\,(e^z)$. The general form of Cauchy integral formula gives the value of this integral as $\dfrac{2\pi i}{3!} f^{(3)}(0) = -\dfrac{2}{3}\pi i$.

Q-3) Find the residues of the following functions at $z = 0$:

a) $f(z) = \dfrac{1}{z^2 \sinh z}$.

b) $g(z) = \dfrac{1}{(2z + 3z^2 + 5z^3)^2}$.

Answers: (a) This is example 2 on page 171. Also studied in exercises 12 and 13 on page 175.

$$\begin{aligned}
\frac{1}{z^2 \sinh z} &= \frac{1}{z^2}\,\frac{1}{z + z^3/3! + z^5/5! + \cdots} \\
&= \frac{1}{z^3}\,\frac{1}{1 + z^2/3! + z^4/5! + \cdots} \\
&= \frac{1}{z^3}\left(1 - \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots\right)
\end{aligned}$$

$$= \frac{1}{z^3} + \frac{-1/6}{z} + \frac{7}{360}z + \cdots$$

from which we see that the residue is $-1/6$.

(b) For this either use exercise 10 on page 198 or do a direct calculation of series and find the residue as -3/4.

If $f(z) = 1/[q(z)]^2$ where z_0 is a simple root of q, then the residue of f at z_0 is given by $-q''(z_0)/[q'(z_0)]^3$. In this problem $q(z) = 2z + 3z^2 + 5z^3$. The residue at 0 is then found to be as $-3/4$.

Or we can do a series calculation as follows, and pick the coefficient of $1/z$:

$$\begin{aligned}
\frac{1}{(2z+3z^2+5z^3)^2} &= \frac{1}{z^2}\,\frac{1}{(2+3z+5z^2)^2} \\
&= \frac{1}{z^2}\,\frac{1}{4+12z+29z^2+30z^3+25z^4} \\
&= \frac{1}{z^2}\,\frac{1}{4(1+3z+(29/4)z^2+(15/4)z^3+(25/4)z^4)} \\
&= \frac{1/4}{z^2}\left(1-3z+\frac{7}{4}z^2+\cdots\right) \\
&= \frac{1/4}{z^2} + \frac{\mathbf{-3/4}}{z} + \frac{7}{16} + \cdots .
\end{aligned}$$

Q-4) Find an upper bound for the modulus of the integral $\int_C \frac{logz}{(z-2)^2}\,dz$, when $logz$ is the branch $|z| > 0, -\pi/2 < argz < 3\pi/2$ and C is the positively oriented circle of radius 1 about $z_0 = 2$.

Answer: Using $|logz| < lnr + |\theta| < ln3 + \pi/6$ one has

$$\left|\int_C \frac{logz}{(z-2)^2}\,dz\right| \le (ln3 + \frac{\pi}{6})2\pi = \frac{\pi^2}{3} + 2\pi ln3,$$

where we used $|z-2| = 1$ on C, and the length of C is 2π.

One can also evaluate this integral using the general form of Cauchy Integral Formula since the integrand is analytic on and inside the circle C. Thus the integral is $2\pi i f'(2) = i\pi$, where $f(z) = logz$. The magnitude is exactly π.

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