1 a) $f(t) = Re\left[F \cdot e^{s \cdot t}\right]$

$= Re\left[F \cdot e^{\sigma \cdot t} \cdot e^{i \cdot w \cdot t}\right] = e^{\sigma \cdot t} \cdot Re\left[F \cdot e^{i \cdot w \cdot t}\right]$

$= e^{\sigma \cdot t} \cdot \left\{ Re[F] \cdot \cos(wt) - Im[F] \cdot \sin(wt) \right\}$

i) $f(t) = \cos\left(10t + \frac{\pi}{6}\right) = \underbrace{Re\left[e^{i \cdot \frac{\pi}{6}} \cdot e^{i \cdot 10 \cdot t}\right]}$

$\sigma = 0$ $\quad F = e^{i \cdot \frac{\pi}{6}} \rightarrow$ phasor

$w = 10$ $\quad s = i \cdot 10 \rightarrow$ complex frequency

ii) $f(t) = 3 \cdot e^{-t} \cdot \sin(5t) = \underbrace{e^{-t} \cdot Re\left[3 \cdot e^{-i \cdot \frac{\pi}{2}} \cdot e^{i \cdot 5 \cdot t}\right]}$

$\sigma = -1$ $\quad F = 3 \cdot e^{-i \cdot \frac{\pi}{2}} \rightarrow$ phasor

$w = 5$ $\quad s = -1 + 5i \rightarrow$ complex frequency

iii) $f(t) = 4 \cdot e^{-3 \cdot t} = \underbrace{e^{-3 \cdot t} \cdot Re\left[4 \cdot \underbrace{e^{i \cdot 0 \cdot t}}_{1}\right]}$

$\sigma = -3$ $\quad F = 4 \rightarrow$ phasor

$w = 0$ $\quad s = -3 \longrightarrow$ complex frequency

b) If $f(t)$ has more than one frequency component, then it won't have a phasor representation. For example if $f(t) = (\cos t) + (\cos 2t)$ then $f(t)$ can't be represented by $\text{Re}\left[F . e^{s.t}\right]$ where F is independent of t. Such functions have more than one complex frequency and as a result they don't have a phasor F which is independent of t. for $f(t) = (\cos t) + (\cos 2t)$;

$$f(t) = (\cos t) + (\cos 2t) = \text{Re}\left[e^{i.t} + e^{i.2.t}\right]$$

$e^{i.t} + e^{i.2.t}$ can't be represented by $F . e^{\sigma . t} . e^{i.w.t}$ where F is independent of t.

$$e^{i.t} + e^{i.2.t} = e^{i.t} . \underbrace{\left[1 + e^{i.t}\right]}$$

$$= e^{i.2.t} . \underbrace{\left[1 + e^{-i.t}\right]}$$

We don't have a F independent of t.

2) $f(z)$ and $f'(z)$ are analytic within and on C. z_0 is not on C.

There are two cases:

i) z_0 is in C. ii) z_0 is outside C.

for case i); $\int_C \frac{f'(z)}{z-z_0} dz = \boxed{2.\pi.i.f'(z_0)}$

Cauchy Integral formula

$\int_C \frac{f(z)}{(z-z_0)^2} . dz = \frac{2.\pi.i}{1!} . (f(z))' \Big|_{z=z_0}$

$= \boxed{2.\pi.i.f'(z_0)}$

Cauchy Integral formula

They are equal.

for case ii) since z_0 is outside C, $\frac{f'(z)}{z-z_0}$ and $\frac{f(z)}{(z-z_0)^2}$ are analytic within and on C.

since C is a simple closed contour, we have: $\int_C \frac{f'(z)}{z-z_0} dz = \int_C \frac{f(z)}{(z-z_0)^2} dz = 0$

So $\int_C \frac{f'(z)}{z-z_0} dz$ and $\int_C \frac{f(z)}{(z-z_0)^2} dz$

are always equal under given conditions.

3) $z_n = (-2) + i \cdot \frac{(-1)^n}{n^2} = x_n + i \cdot y_n$

$x_n = -2 \quad y_n = \frac{(-1)^n}{n^2}$

$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \cdot \lim_{n\to\infty} y_n$

$\lim_{n\to\infty} x_n = -2 \quad \lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0$

So; $\lim_{n\to\infty} z_n = -2 + i \cdot 0 = -2$

We will also show that z_n converges to -2 by using the definition of limit.

If z_n converges to (-2); then for every positive number ε, there should exist a positive integer such that

$|z_n - (-2)| < \varepsilon$ whenever $n > n_0$.

$|z_n - (-2)| = \left| i \cdot \frac{(-1)^n}{n^2} \right| = \frac{1}{n^2}$

$|z_n - (-2)| < \varepsilon \rightarrow \frac{1}{n^2} < \varepsilon \Rightarrow n > \frac{1}{\sqrt{\varepsilon}}$

So if we choose n_0 as the biggest integer which is smaller than or equal to $\frac{1}{\sqrt{\varepsilon}}$, the desired condition is satisfied. This means that we can find a n_0 for every positive ε value. This concludes the proof.

So $\lim_{n\to\infty} z_n = -2$.

4) $\dfrac{1}{4z-z^2} = \dfrac{1}{4z} \cdot \dfrac{1}{1-\frac{z}{4}}$

$\dfrac{1}{1-\frac{z}{4}} = \displaystyle\sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n$ when $\underbrace{\left|\frac{z}{4}\right| < 1}_{|z|<4}$.

So when $|z| < 4$, $\dfrac{1}{1-\frac{z}{4}} = \displaystyle\sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n$

But $\dfrac{1}{4z-z^2} = \dfrac{1}{4z} \cdot \dfrac{1}{1-\frac{z}{4}}$ is singular at $z=0$.

So when $0 < |z| < 4$;

$$\frac{1}{4z-z^2} = \frac{1}{4z} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$$

$$= \sum_{n=-1}^{\infty} \frac{z^n}{4^{n+2}} = \frac{1}{4 \cdot z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$$

5) Since $f(z)$ is entire, it satisfies Cauchy's inequality for every z_0 and R value. We should find M_R.

On $C: z = z_0 + R.e^{i.\theta}$, $-\pi \leq \theta \leq \pi$

$$|z| = |z_0 + R.e^{i.\theta}| \leq |z_0| + R$$

$$|f(z)| \leq A.|z|^3 \leq \underbrace{A.(|z_0|+R)^3}_{M_R}$$

So for every z_0 and R value,

$$|f^{(n)}(z_0)| \leq \frac{n!.A.(|z_0|+R)^3}{R^n} \qquad n = 1, 2, 3, \ldots$$

for $n=4$, $|f^{(4)}(z_0)| \leq \dfrac{24.A.(|z_0|+R)^3}{R^4}$

If we let $R \to \infty$, $|f^{(4)}(z_0)| \leq 0$ for every z_0 value.

This means that $f^{(4)}(z_0) = 0$ for every z_0 value.

So $f^{(4)}(z) = 0. \longrightarrow f^{(3)}(z) = c \rightarrow$ constant

$f^{(2)}(z) = cz + d \rightarrow$ constant

$$|f^{(2)}(z_0)| \leq \frac{2 \cdot A \cdot (|z_0|+R)^3}{R^2}$$

We also know $f^{(2)}(z) = cz+d$.

If we let $z_0 = 0$, we get:

$$|\underbrace{f^{(2)}(0)}_{d}| = |d| \leq \frac{2 \cdot A \cdot R^3}{R^2} \leq 2 \cdot A \cdot R$$ for every possible R value.

If we let $R \to 0$, we get: $\underbrace{|d| \leq 0}_{\downarrow \atop d=0}$

So $f^{(2)}(z) = c \cdot z \longrightarrow f^{(1)}(z) = \frac{c}{2} \cdot z^2 + \underbrace{e}_{\text{constant}}$

$$|f^{(1)}(z_0)| \leq \frac{A \cdot (|z_0|+R)^3}{R}$$

If we let $z_0 = 0$, we get:

$|\underbrace{f^{(1)}(0)}_{e}| = |e| \leq A \cdot R^2$ for every possible R value.

If we let $R \to 0$, we get: $\underbrace{|e| \leq 0}_{e=0}$

$$f^{(1)}(z) = \frac{c}{2} \cdot z^2 \rightarrow f(z) = \frac{c}{6} \cdot z^3 + h \longrightarrow \text{constant}$$

$f(0) = h$ We know $|f(z)| \leq A \cdot |z|^3$

$|f(0)| = |h| \leq A \cdot 0 \to \underbrace{|h| \leq 0}_{h = 0}$

So $f(z) = \frac{c}{6} \cdot z^3$

$= a_1 \cdot z^3$

where c and a_1 are

constants.