MATH 206 HW7

1) Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains and specify those domains.

2)(a) Let $f(z)$ denote a function which is analytic in some annular domain about the origin that includes the unit circle $z = e^{i\phi}(-\pi \le \phi \le \pi)$. By taking that circle as the path of integration in expressions

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n = 0, 1, 2, \ldots)$$

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz \qquad (n = 1, 2, \ldots)$$

for the coefficients a_n and b_n in a Laurent series in powers of z, show that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right]d\phi$$

when z is any point in the annular domain.

(b) Write $u(\theta) = \operatorname{Re}[f(e^{i\theta})]$, and show how it follows from the expansion in part (a) that

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos\left[n(\theta-\phi)\right]d\phi\,.$$

3) If C denotes the circle $|z| = 1$, taken counterclockwise, then show that

$$\int_C \exp\left(z + \frac{1}{z}\right)dz = 2\pi i\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}\,.$$

4) The *Euler numbers* are the numbers $E_n (n = 0, 1, 2, \ldots)$ in the Maclaurin series representation

$$\frac{1}{\cosh(z)} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n \quad (|z| < \pi/2).$$

Point out why this representation is valid in the indicated disk and why $E_{2n+1} = 0 \ (n = 0, 1, 2, \ldots)$. Then show that

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad \text{and} \quad E_6 = -61.$$

5) Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, then the function g defined by the equations

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^{m+1}} & \text{when } z \neq z_0 \\ \\ \frac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when } z = z_0 \end{cases}$$

is analytic at z_0.