Date: January 13, 2009, Tuesday NAME:...
Time: 9:30-11:30
Altıntaş & Sertöz STUDENT NO:...

Math 206 Complex Calculus – Final Exam

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 4 questions on your exam booklet. Write your name on the top of every page. A correct answer without proper reasoning may not get any credit.

Q-1) The function $f(z) = \dfrac{e^{1/z}}{1+z^2}$ has the Laurent expansion

$$f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \frac{a_4}{z^4} + \frac{a_5}{z^5} + \frac{a_6}{z^6} + \cdots$$

valid for $|z| > 1$. It is easy to see that $a_0 = a_1 = 0$. Find a_2, a_3, a_4, a_5, a_6. (5 points each!)

Solution:

$$\begin{aligned}
\frac{e^{1/z}}{1+z^2} &= e^{1/z}\,\frac{1}{z^2}\,\frac{1}{1+1/z^2} \\
&= \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \frac{1}{24z^4} + \cdots\right)\left(\frac{1}{z^2}\right)\left(1 - \frac{1}{z^2} + \frac{1}{z^4} + \cdots\right) \\
&= \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \frac{1}{24z^4} + \cdots\right)\left(\frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} + \cdots\right) \\
&= \frac{1}{z^2} + \frac{1}{z^3} + (-1 + \frac{1}{2})\frac{1}{z^4} + (-1 + \frac{1}{6})\frac{1}{z^5} + (1 - \frac{1}{2} + \frac{1}{24})\frac{1}{z^6} + \cdots
\end{aligned}$$

which gives $a_2 = 1$, $a_3 = 1$, $a_4 = -\dfrac{1}{2}$, $a_5 = -\dfrac{5}{6}$, $a_6 = \dfrac{13}{24}$.

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Q-2) Evaluate the real improper integral

$$\int_0^\infty \frac{\ln x}{(1+x^2)^2}\,dx.$$

(Observe that this is a real integral and the answer must be a real number!)

Solution: Let $f(z) = \dfrac{\log z}{(1+z^2)^2}$ and let $\gamma_{R,\rho} = [-R,-\rho] + C_\rho + [\rho, R] + C_R$, where $0 < \rho < 1 < R$, C_ρ is the semicircle of radius ρ centered at the origin and lying in the upper half plane, traversed in the clockwise direction, C_R is the semicircle with radius R centered at the origin, lying in the upper half plane and traversed in the counterclockwise direction. By the Cauchy Residue Theorem, the value of the integral of $f(z)$ along the contour $\gamma_{R,\rho}$ is $2\pi i$ times the sum of the residues of f inside the contour. The function f has a double singularity at $z = i$ and its residue there is $\phi\,'(i)$ where $\phi(z) = \dfrac{logz}{(z+i)^2}$.

We calculate that the real part of $2\pi i\phi\,'(i)$ is $-\dfrac{\pi}{2}$.

It is easy to show, using triangle inequalities, that the limits of the integrals of f on C_ρ and C_R as $\rho \to 0$ and $R \to \infty$ are each zero.

The integral of f on $[-R,-\rho]$ and on $[\rho, R]$, after taking the above limits, converges to

$$2\int_0^\infty \frac{\ln x}{(1+x^2)^2}\,dx + i\cdot(\text{something}).$$

Hence we find, using the above calculated residue that

$$\int_0^\infty \frac{\ln x}{(1+x^2)^2}\,dx = -\frac{\pi}{4}.$$

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Q-3) Use the technique of Laplace transform to solve the following differential equation

$$y'' + 3y' + 2y = e^t, \quad y(0) = a, \;\; y'(0) = b,$$

where a and b are some real constants.

Solution: If we denote by $Y(s)$ the Laplace transform of $y(t)$, then we find that

$$\begin{aligned} Y(s) &= \frac{as^2 + (2a+b)s + (1-3a-b)}{(s-1)(s+1)(s+2)} \\ &= \frac{1}{6}\,\frac{1}{s-1} + \left(-\frac{1}{2} + 2a + b\right)\frac{1}{s+1} + \left(\frac{1}{3} - a - b\right)\,\frac{1}{s+2} \end{aligned}$$

and

$$y(t) = \frac{1}{6}\,e^t + \left(-\frac{1}{2} + 2a + b\right)e^{-t}\left(\frac{1}{3} - a - b\right)\,e^{-2t}.$$

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Q-4) Find all roots of the following equation

$$\cos z = 25$$

Hint:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

Solution:

$$\cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy)$$

But noting that

$$\cos(iy) = \cosh y$$

and

$$\sin(iy) = i \sinh y$$

we can write the equation as:

$$\cos x \cosh y - i \sin x \sinh y = 25.$$

Equating the imaginary parts we get either $y = 0$, or $x = n\pi$. However, considering the real parts, we see that $y = 0$ has no solution. Since the real part has to be positive, we should have $x = 2n\pi$. Finally, $\cosh y = 25$ yields $y = \cosh^{-1} 25$. Hence, the result is $z = 2n\pi + i \cosh^{-1} 25$.