Math 214 Advanced Calculus II
Midterm Exam II
SOLUTIONS

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1) Suppose V is open in $\mathbb{R}^n$ and $f : V \to \mathbb{R}$ is continuously differentiable with $\Delta_f \neq 0$ on V. Prove that

$$\lim_{r \to 0} \frac{\mathrm{Vol}(f(B_r(\mathbf{x}_0)))}{\mathrm{Vol}(B_r(\mathbf{x}_0))} = |\Delta_f(\mathbf{x}_0)|$$

for every $\mathbf{x}_0 \in V$.

Solution:

Fix an arbitrary $\mathbf{x}_0 \in V$. Note that $\overline{B_r(\mathbf{x}_0)}$ is compact, Δ_f attains its *inf* and *sup* there, and

$$\begin{aligned}
\mathrm{Vol}(f(B_r(\mathbf{x}_0))) &= \int_{f(B_r(\mathbf{x}_0))} d\mathbf{u} && (1)\\
&= \int_{B_r(\mathbf{x}_0)} |\Delta_f(\mathbf{x})|\, d\mathbf{x} && (2)\\
&= c\,\mathrm{Vol}(B_r(\mathbf{x}_0)) && (3)\\
&= |\Delta_f(\mathbf{x}_1)|\,\mathrm{Vol}(B_r(\mathbf{x}_0)) && (4)
\end{aligned}$$

where
(1) follows from Cor. 7.5 (p349),
(2) follows from Thm. 7.14 (p375),
(3) holds for some c between the *inf* and *sup* of Δ_f on $B_r(\mathbf{x}_0)$, Thm. 7.9 (p354),
(4) holds for some $\mathbf{x}_1 \in \overline{B_r(\mathbf{x}_0)}$ since $|\Delta_f(\mathbf{x})|$ is continuous there.
Finally taking the limit as $r \to 0$ gives the required result.
Note that this is Exercise 7 on page 382. (The problem would be more meaningful if the range of f is $\mathbb{R}^n$)...!

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2) Suppose V is open in $\mathbb{R}^n$ and $f : V \to \mathbb{R}$ is continuous. Prove that if

$$\int_E f(\mathbf{x})d\mathbf{x} = 0$$

for all Jordan regions $E \subset V$, then $f = 0$ on V.

Solution: Assume that there is an $\mathbf{x}_0 \in V$ where $f(\mathbf{x}_0) = 2\epsilon > 0$.
Let $U = \{\mathbf{x} \in V \mid f(\mathbf{x}) > \epsilon > 0\ \}$. We know that U is not empty since $\mathbf{x}_0 \in U$. Since f is continuous, U is open. (U is the preimage of the open interval (ϵ, ∞)). By continuity of f there is an open ball B_{2r} of radius $2r$ around $\mathbf{x}_0$, where f is strictly positive. The integral of f on B_r is zero by the assumption of the theorem. (Note that B_r is a Jordan region). By the mean value theorem this is equal to $c\ Vol(B_r)$ for some c between the inf and sup of f on B_r. Since f is continuous on B_{2r}, there is a point $\mathbf{x}_1 \in \overline{B_r}$ where $f(\mathbf{x}_1) = c$, so $c \neq 0$. But this forces the volume of B_r to be zero, a contradiction. So there is no point where f is strictly positive. Similarly f cannot take a strictly negative value. So f is identically zero on V.
Note that this is Exercise 9 on page 357.

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3) Let S be the surface defined by $z = 7\sqrt{x^2 + y^2}$, $7 \leq z \leq 28$. Evaluate the integral

$$\iint_S F \cdot \vec{\mathbf{n}}\ d\sigma$$

where $F(x, y, z) = (x^2, y(1 - 2x), -z)$ and $\vec{\mathbf{n}}$ is the unit normal vector pointing outward.

Solution: Evaluating this integral directly requires a lot of tedious work, so we try using Stokes' theorem which reduces this integral to an integral of the boundary. Since the boundary consists of two circles, we expect that the resulting integral will be easy. But to apply Stokes' theorem we must express F as curl$G$. Since div$F = 0$, such a G exists, see Theorem 8.6 on page 450. Set $G = (P, Q, R)$. Trial and error, with some luck, will give you a G with curl$G = F$. For example I set $Q = 0$ and tried to solve the remaining equalities:
$R_y = x^2$, $P_z - R_x = y - 2xy$, $-P_y = -z$. This easily yields a solution, for example $G = (yz, 0, yx^2)$. Now Stokes' theorem says that the required integral is equal to $\int_{\partial S} G \cdot T\ ds$.
We now parametrize the boundary $\partial S = C_1 - C_2$.
C_1: $\phi(t) = (\cos t, \sin t, 7)$, $t \in [0, 2\pi]$,
C_2: $\psi(t) = (4\cos t, 4\sin t, 28)$, $t \in [0, 2\pi]$.
$G(\phi(t)) \cdot \phi'(t)dt = -7\sin^2 t\ dt$.
$G(\psi(t)) \cdot \psi'(t)dt = -448\sin^2 t\ dt$.
So we integrate $441\sin^2 t dt$ from 0 to 2π to obtain 441π as the answer.

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4) Find the surface area of the cap $x^2 + y^2 + z^2 = R^2$, $z \geq \sqrt{R^2 - A^2}$, where $R \geq A \geq 0$.

Solution: We first parametrize the surface:
$\phi(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{R^2 - r^2})$, $(r, \theta) \in [0, A] \times [0, 2\pi] = E$.
To find the surface area we need to evaluate $\int_E \parallel N_{\phi_r \times \phi_\theta} \parallel d(r, \theta)$, see Definition 8.12 on page 424. A straightforward calculation gives $\parallel N_{\phi_r \times \phi_\theta} \parallel = r\left(R^2/(R^2 - r^2)\right)^{1/2}$. Integrating this over E gives $2\pi R^2 - 2\pi R\sqrt{R^2 - A^2}$.

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5) **(i)** Construct a C^∞ vector field $F : \mathbb{R}^3 \backslash \{(0,0,0)\} \to \mathbb{R}^3$ such that F is not defined at the origin and $\mathrm{div}F = 0$.
(ii) Let E_1 be the surface given by $x^2 + y^2 + z^2 = 1$ and E_2 the surface given by $x^2/25 + y^2/36 + z^2/49 = 1$. Show that

$$\iint_{E_1} F \cdot \vec{\mathbf{n}}_1 \, d\sigma = \iint_{E_2} F \cdot \vec{\mathbf{n}}_2 \, d\sigma$$

where F is the vector field you found in the first part, and $\vec{\mathbf{n}}_i$ is the unit normal pointing outward of the surface E_i, $i = 1, 2$.

Solution: The easiest way to construct a vector field F with zero divergence is to start with a vector field G and set F to be $\mathrm{curl}G$. Since we want F to be defined everywhere except at the origin we can start with an arbitrary C^∞ vector field G which is defined everywhere except at the origin. One such easy example is $G = (\log(x^2 + y^2 + z^2), \log(x^2 + y^2 + z^2), \log(x^2 + y^2 + z^2))$. To prove the second part let the region between these surfaces be denoted by E. Then use Gauss' theorem, Theorem 8.4 on page 441, and observe that the right hand side is zero since $\mathrm{div}F = 0$. This gives, after observing correct orientations of the surfaces E_1 and E_2, that the required equality holds.
Note that this is Exercise 10.c on page 454. I solved parts (a) and (b) in class and promised to ask part (c) in the exam!
