Date: November 20, 2006, Monday

Math 302 Complex Calculus – Midterm Exam II – Solutions

Q-1) Find, if it exists, an entire function which vanishes only at the positive primes.

Solution: Since $\sum \frac{1}{p}$ diverges but $\sum \frac{1}{p^2}$ converges, where the sum is over all primes, we have that

$$\prod \left[\left(1 - \frac{z}{p}\right) e^{z/p} \right]$$

is an entire function vanishing at only the primes. Here the product is over all the primes.

Q-2) Find a C-harmonic function in the unit disk with boundary values $x^3 - xy$.

Solution: Since $z^3 = (x^3 - 3xy^2) + i(3x^2 - y^3)$ is analytic on the unit disk, its real part is harmonic there with boundary values $x^3 - 3xy^2 = 4x^3 - 3x$. Clearly xy is harmonic everywhere. Therefore the required function is

$$u(x, y) = \frac{1}{4}(x^3 - 3xy^2) + \frac{3}{4}x - xy.$$

Q-3) Describe the image of the unit disk under the map $f(z) = \dfrac{1}{z - \alpha}$, where $\alpha > 1$.

Solution: f is a conformal map sending circles to circles. In particular since $\alpha > 1$, the unit circle is mapped to another circle with center c_0 on the real line and with radius r_0. By symmetry $f(-1) - f(1) = 2r_0$ and $f(-1) + r_0 = c_0$. These give $r_0 = \dfrac{1}{\alpha^2 - 1}$ and $c_0 = \dfrac{-\alpha}{\alpha^2 - 1}$.

Q-4) Find $\displaystyle\sum_{n=0}^{\infty} \binom{2n}{n+1} \frac{1}{5^n}$. (Here take $\binom{0}{1} = 0$.)

Solution: Let C be the circle $|z| = 1$. Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{2n}{n+1} \frac{1}{5^n} &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C \frac{(1+z)^{2n}}{(5z)^n} \frac{dz}{z^2} \\
&= \frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} \left[\frac{(1+z)^2}{(5z)} \right]^n \frac{dz}{z^2} \\
&= \frac{1}{2\pi i} \int_C f(z)\, dz
\end{aligned}$$

where $f(z) = \dfrac{5z}{z(z^2-3z+1)}$. Its poles inside C are $z_1 = 0$ and $z_2 = (3-\sqrt{5})/2$. Sum of the residues at these poles gives the total sum as $(3\sqrt{5}-5)/2 \approx 0.8541$. Check that $\left|\dfrac{(1+z)^2}{5z}\right| \leq \dfrac{4}{5} < 1$ which gives uniform convergence to justify the interchange of infinite sum and the integral.

Q-5) Use residue theory to evaluate $\displaystyle\int_0^\infty \frac{x^2}{1+x^4}\,dx$.

Solution: Let γ be the path $[-R,R]+C_R$ where C_R is the semicircle of radius R lying in the upper half plane. Integrating the function $f(z) = \dfrac{z^2}{1+z^4}$ over this path with $R > 1$ and using the residue theorem we find that the value of the integral is $(2\pi i)\left(\dfrac{\sqrt{2}-i\sqrt{2}}{8} + \dfrac{-\sqrt{2}-i\sqrt{2}}{8}\right) = \dfrac{2\sqrt{2}\pi}{4}$, where we have added the residues at $z = exp(\pi i/4)$ and at $z = exp(3\pi i/4)$. Taking the limit as R goes to infinity and observing that f is an even function we find

$$\int_0^\infty \frac{x^2}{1+x^4}\,dx = \frac{\sqrt{2}\pi}{4} \approx 1.11.$$
