Due Date: June 17, 2011 Friday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 302 Complex Analysis II – Homework 2

1	2	TOTAL
10	10	20

Please do not write anything inside the above boxes!

Check that there are 2 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Discuss the convergence of $\sum_{n=0}^{\infty} \binom{2n+1}{n} x^n$, where x is a real number. Find the sum when it exists.

Solution:

Let $a_n(x) = \binom{2n+1}{n} x^n$. The ratio test gives

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{4n+6}{n+2}\,|x| \to 4|x| \text{ as } n \to \infty.$$

Therefore the series converges absolutely for $|x| < 1/4$. To check convergence for $x = 1/4$ we use Raabe's test.

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}(1/4)}{a_n(1/4)}\right) = \lim_{n\to\infty} \frac{2n}{4n+8} = \frac{1}{2} < 1,$$

so the series diverges for $x = 1/4$.

Next we check convergence at $x = -1/4$. In this case $a_n(-1/4) = (-1)^n a_n(1/4)$. Check that

$$\frac{a_{n+1}(1/4)}{a_n(1/4)} = \frac{2n+3}{2n+4} < 1,$$

so

$$a_{n+1}(1/4) < a_n(1/4),\ n = 0, 1, \ldots$$

and thus $a_n(1/4)$ strictly decreases as $n \to \infty$.

To show that $a_n(1/4) \to 0$ as $n \to \infty$, we use Stirling's theorem which says that

$$\lim_{n\to\infty} \frac{n!\,e^n}{\sqrt{2\pi}\,n^n\,n^{1/2}} = 1.$$

Let $S(n) = \frac{n!\,e^n}{\sqrt{2\pi}\,n^n\,n^{1/2}}$. Then $n! = S(n)\left[\frac{\sqrt{2\pi}\,n^n\,n^{1/2}}{e^n}\right]$. We use this in checking the behavior of $a_n(1/4)$ as $n \to \infty$.

$$\begin{aligned} a_n(1/4) &= \frac{(2n+1)!}{n!(n+1)!}\,\frac{1}{4^n} \\ &= \frac{S(2n+1)}{S(n)S(n+1)}\left[\sqrt{\frac{2}{\pi}}\left(1+\frac{1/2}{n}\right)^n\left(1-\frac{1/2}{n+1}\right)^{n+1}\left(2-\frac{1}{n+1}\right)^{1/2}\right]\left(\frac{1}{n}\right)^{1/2}, \end{aligned}$$

which shows that $\lim_{n\to\infty} a_n(1/4) = 0$. We then conclude that the series also converges for $x = -1/4$ by the alternating series test.

Hence the interval of convergence of the series is $-\frac{1}{4} \le x < \frac{1}{4}$.

To calculate the series, we start with the usual observation that for any $R > 0$,

$$\binom{2n+1}{n} = \frac{1}{2\pi i}\int_{|z|=R} \frac{(z+1)^{2n+1}}{z^{n+1}}\,dz.$$

Then

$$\begin{aligned}
\sum_{n=0}^{\infty}\binom{2n+1}{n}x^n &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{|z|=R}\frac{(z+1)^{2n+1}x^n}{z^{n+1}}\,dz \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{|z|=R}\left[\frac{(1+z)^2x}{z}\right]^n\frac{1+z}{z}\,dz \\
&= \frac{1}{2\pi i}\int_{|z|=R}\sum_{n=0}^{\infty}\left[\frac{(1+z)^2x}{z}\right]^n\frac{1+z}{z}\,dz \\
&= \frac{1}{2\pi i}\int_{|z|=R}\frac{1}{1-\left[\frac{(1+z)^2x}{z}\right]}\frac{1+z}{z}\,dz \\
&= \frac{1}{2\pi i}\int_{|z|=R}\frac{z+1}{z-(1+z)^2x}\,dz.
\end{aligned}$$

The change of infinite sum and integration is justified when the convergence of the infinite sum is uniform on the given circle. We check that for $|z|=1$,

$$\left|\frac{(1+z)^2x}{z}\right| \le 4|x| < 1 \text{ when } |x| < \frac{1}{4},$$

so we can take $R=1$ and $|x|<1/4$.

Now we evaluate the integral using residue theory. The roots of $z-(1+z)^2x=0$ are

$$c_1 = \frac{1-2x-\sqrt{1-4x}}{2x} \text{ and } c_2 = \frac{1-2x+\sqrt{1-4x}}{2x}.$$

Check that $0<c_1<1<c_2$. We need the residue at c_1 since it is the only pole inside the unit circle.

$$\mathrm{Res}\left(\frac{(1+z)^2x}{z}, c_1\right) = \frac{1-\sqrt{1-4x}}{2x\sqrt{1-4x}}.$$

Hence we find

$$\sum_{n=0}^{\infty}\binom{2n+1}{n}x^n = \frac{1-\sqrt{1-4x}}{2x\sqrt{1-4x}} \text{ for } -1/4 \le x < 1/4,$$

where the equality of x to $-1/4$ is given by Abel's theorem. Notice that the formula makes sense when $x=0$ also, since

$$\frac{1-\sqrt{1-4x}}{2x} = 1+x+2x^2+5x^3+\cdots \text{ around } x=0.$$

Algebraically speaking,

$$\frac{1-\sqrt{1-4x}}{2x} = \frac{1-\sqrt{1-4x}}{2x}\cdot\frac{1+\sqrt{1-4x}}{1+\sqrt{1-4x}} = \frac{2}{\sqrt{1-4x}(1+\sqrt{1-4x})}.$$

The power of this summation formula lies in the fact that the convergence is extremely slow as $|x|$ is close to $1/4$. For example when $x=-1/4$, the formula gives the sum as $0.5857...$ but a direct summation by computer fails to find the second digit after the decimal point even after $n=17,000$. In fact the third digit is not yet stabilized even after $5,000,000$ iterations.

NAME: STUDENT NO:

Q-2) Find the sum of $\sum_{n=0}^{\infty} \frac{1}{n^4+1}$. In general describe how to find $\sum_{n=0}^{\infty} \frac{1}{n^{2k}+1}$, where k is a positive integer.

Solution:

$$\sum_{n=0}^{\infty} \frac{1}{n^{2k}+1} = -\frac{1}{2} \sum_{j=0}^{2k} \operatorname{Res}\left(\frac{\pi \cot \pi z}{1+z^{2k}}, z_j\right),$$

where $z_0 = 0$ and $z_1, \ldots, z_{2k}$ are the roots of $1 + z^{2k} = 0$. In particular

$$\sum_{n=0}^{\infty} \frac{1}{n^4+1} = -\frac{1}{2} \sum_{j=0}^{4} \operatorname{Res}\left(\frac{\pi \cot \pi z}{1+z^4}, z_j\right) = 0.5784...$$