Due Date: June 30, 2011 Thursday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 302 Complex Analysis II – Homework 5 – Solutions

1	2	TOTAL
10	10	20

Please do not write anything inside the above boxes!

Check that there are 2 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Show that $e^{e^z} \ll 1$ throughout the boundary of the region

$$D = \{x + iy \mid -\frac{\pi}{2} < y < \frac{\pi}{2}\}.$$

Show that it is the "smallest" such analytic function.

This is Exercise 3 of your textbook at the end of the chapter on Maximum-Modulus Theorems. (page 199 on 2nd edition, page 223 on 3rd edition.)

Explain what the problem is asking for. Give your solution in detail. There is a clue given at the back of the book which does not count as a solution!

Solution: Since e^z maps the boundary of D onto the imaginary axis, $\alpha(z) = e^{e^z}$ has modulus 1 on the boundary of D.

Note that $\alpha(\log z) = e^z$ and is unbounded on D.

Let $f(z)$ be a function"smaller" that $\alpha(z)$ on D, in the sense that for all $\epsilon > 0$ there exists a constant A_ϵ such that $|f(z)| \leq A_\epsilon |e^{\epsilon e^z}|$ for $z \in D$, then $|f(\log z)| \leq A_\epsilon e^{\epsilon |z|}$.

It follows that $f(z)$ is bounded on ∂D as well as on D, by Phragmén-Lindelöf theorem.

Consider the collection of all functions F analytic on $\bar{D}$, satisfying the condition that $F(z)$ is bounded on ∂D but $F(\log z)$ is unbounded in D. Then in the above sense, $\alpha(z) = e^{e^z}$ is the "smallest" function of this collection.

NAME: STUDENT NO:

Q-2) We proved the following theorem: If f is a non-constant entire function, there exists a curve along which f approaches infinity.

(i) Explain why Liouville's theorem does not immediately guarantee the existence of such a curve. Does there exist a non-constant entire function f and a path γ such that f does not go to infinity along γ even though $\gamma(t) \to \infty$ as $t \to \infty$? If *yes*, give an example, if *no*, explain why.

(ii) In the proof of the above theorem we explicitly used the polygonally connectedness of certain open sets which were connected components of some crucially defines sets. Prove that these sets were actually polygonally connected. (i.e., using the notation of the book, show that each S_k is polygonally connected.)

Solution:

(i) Liouville's theorem assures the existence of a sequence $z_1, z_2, \ldots$ of complex numbers converging to infinity such that the sequence $f(z_1), f(z_2), \ldots,$ converges to infinity. But the points z_k may be connected by a path such that f along that path need not converge to infinity. Here is an example.

Define $f(z) = z\sin^2(\pi z/2)$. Let $z_k = 2k+1$ and take the real line as the path γ joining them. Then clearly $f(k) = 2k+1$ and goes to infinity as k goes to infinity but each $w_k = 2k$ is also on the curve γ with $w_k < z_k < w_{k+1}$ and $f(w_k) = 0$, so $f(z)$ does not converge to infinity along γ.

(ii) Let S be an open and connected set. Fix a point $p \in S$. Let A be the set of points in S which can be polygonally connected to p. A is clearly not empty since $p \in A$. Moreover A is clearly open since if $q \in A$, then there is an open ball U around q totally lying in S, since S is open. But all the points in this ball can be polygonally connected to q, so U is in A. Similarly the complement of A in S is open, so A is closed in S. Thus A is a non-empty subset of S which is both open and closed. So A is a component of S, but S is connected, so $S = A$, which says that all points of S can be polygonally connected to p, and hence to each other.