Due Date: July 21, 2011 Thursday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 302 Complex Analysis II – Homework 8 – Solutions

1	2	TOTAL
10	10	20

Please do not write anything inside the above boxes!

Check that there are 2 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. **A correct answer without proper or with too much reasoning may not get any credit.**

NAME: STUDENT NO:

Q-1) While trying to extend the Gamma function to the whole plane we made use of the following function

$$f(z) = \frac{1}{z} - \frac{1}{(z+1)} + \frac{1}{2!(z+2)} - \cdots + \frac{(-1)^n}{n!(z+n)} + \cdots .$$

We claimed that "*$f(z)$ is an analytic function for all $z \in \mathbb{C}$ except when $z = 0, -1, -2, \ldots$.*" Prove this claim.

Solution:

Let $\sigma_n(z) = \dfrac{(-1)^n}{n!(z+n)}$. Let D be any compact region in the plane not including any of the points $z = 0, -1, -2, \ldots$. Let $\delta = \inf\{|z-k| \mid z \in D \text{ and } k = 0, -1, -2, \ldots\}$.

Since D is bounded, it is included in a disk of radius $R > 0$ around the origin. There are only finitely many integers of the form $0, -1, -2, \ldots$ inside this disk. Around each such integer there is an open disk not intersecting D since D is closed. The smallest of these finitely many positive radii is δ, hence $\delta > 0$.

Let $\epsilon > 0$ be given. For any $z \in D$, $|\sigma_n(z)| = \dfrac{1}{n!|z+n|} \leq \dfrac{1}{n!\delta}$. Since $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!\delta}$ converges, by Weierstrass M-test, the convergence of $\displaystyle\sum_{n=0}^{\infty} \sigma_n(z)$ to $f(z)$ is uniform.

We showed that $\displaystyle\sum_{n=0}^{\infty} \sigma_n(z)$ converges uniformly to $f(z)$ on compacta. Since each $\sigma_n(z)$ is analytic, except at $z = 0, -1, -2, \ldots$, so is $f(z)$.

NAME: STUDENT NO:

Q-2) Prove that $\displaystyle\sum_{\substack{p:\text{ prime}\\ n\geq 2}} \frac{1}{np^{nz}}$ is analytic in $\operatorname{Re} z > \frac{1}{2}$.

Show your work in detail, explain all your arguments.

Solution: Any finite sum of this expression is an entire function. It remains to show that the infinite sum converges uniformly on compacta for $\operatorname{Re} z > 1/2$. Let $z = x + iy$ and $x > 1/2$.

$$\begin{aligned}
\sum_{n=2}^{\infty} \left| \frac{1}{np^{nz}} \right| &= \sum_{n=2}^{\infty} \frac{1}{np^{nx}} \\
&\leq \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{p^{nx}} \\
&= \frac{1}{2} \left(\frac{1}{1 - 1/p^x} - 1 - \frac{1}{p^x} \right) \\
&= \frac{1}{2p^{2x}} \frac{p^x}{p^x - 1} \\
&= \frac{1}{2p^{2x}} \left(1 + \frac{1}{p^x - 1} \right) \\
&\leq \frac{1}{2p^{2x}} \left(1 + \frac{1}{2^{1/2} - 1} \right) \\
&= \frac{1}{2p^{2x}} (3.4141\ldots) \\
&< \frac{2}{p^{2x}}.
\end{aligned}$$

Let D be any compact subset of $\operatorname{Re} z > 1/2$. There is a $\delta > 0$ such that for each $z \in D$, $x \geq 1/2 + \delta$. Then $\frac{2}{p^{2x}} \leq \frac{2}{p^{1+2\delta}}$. Since $\displaystyle\sum_{p:\text{ prime}} \frac{2}{p^{1+2\delta}}$ converges, by Weierstrass M-test, $\displaystyle\sum_{\substack{p:\text{ prime}\\ n\geq 2}} \frac{1}{np^{nz}}$ converges uniformly. This completes the proof.