Due Date: November 29, 2013 Friday NAME:..

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Math 302 Complex Analysis II – Homework 2 – Solutions

1	2	3	4	TOTAL
10	10	10	10	40

Please do not write anything inside the above boxes!

Check that there are 3 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

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Q-1) Let L be a line in the complex plane and let T be a Mobius transformation sending L again to a line. Classify all such T.

Solution:

There are two cases. Either $T(\infty) = \infty$, or $T(\infty) \in \mathbb{C}$.

If $T(\infty) = \infty$, then clearly T is linear.

If, on the other hand, $T(\infty) = w_0 \in \mathbb{C}$, then there must exist a $z_0 \in L \subset \mathbb{C}$ such that $T(z_0) = \infty$. Then we must have

$$T(z) = w_0 + \frac{w}{z - z_0},$$

for some $w \in \mathbb{C}$. Let $z_1 \in \mathbb{C}$ be a point on L other than z_0, and let $T(z_1) = w_1 \in T(L)$. Then

$$T(z) = w_0 + \frac{(z_1 - z_0)(w_1 - w_0)}{z - z_0}.$$

Any point on L is of the form $z_0 + t(z_1 - z_0)$ where $t \in \mathbb{R}$. Check that

$$T(z_0 + t(z_1 - z_0)) = w_0 + \frac{1}{t}(w_1 - w_0).$$

So the image is the line through w_0 and w_1.

Conclusion:

If $T(\infty) = \infty$, then let z_0 and z_1 be two different points on L. Let $w_0 = T(z_0)$ and $w_1 = T(z_1)$. Then

$$T(z) = \frac{w_1 - w_0}{z_1 - z_0}\,(z - z_0) + w_0.$$

If $T(\infty) = w_0 \in \mathbb{C}$, then let $z_0 \in L$ be such that $T(z_0) = \infty$. Choose any point $z_1 \in \mathbb{C}$ on L different than z_0 and set $w_1 = T(z_1)$. Then

$$T(z) = w_0 + \frac{(z_1 - z_0)(w_1 - w_0)}{z - z_0}.$$

Another solution which I learned from your papers is the following.

Any Mobius transformation is of the form $T(z) = \frac{az+b}{cz+d}$. If $c = 0$, then T is linear and sends L to a line. If $c \neq 0$, then $T(z) = (f_3 \circ f_2 \circ f_1)(z)$ where

$$f_1(z) = cz + d, \quad f_2(z) = \frac{1}{z}, \quad f_3(z) = \frac{a}{c} - \left(\frac{ad - bc}{c}\right) z.$$

Here f_1 and f_3 are linear and take lines to lines. But f_2 takes only those lines through the origin to lines. If we want T to send L to a line, then there must exist a point $z \in \mathbb{C}$ on L such that $cz + d = 0$.

Conclusion: The Mobius transformations sending L to a line are of the following form.

$$\{\frac{az+b}{cz+d} \mid ad - bc \neq 0 \text{ and } -\frac{d}{c} \in L\}.$$

Note that when $c = 0$, we can interpret $-d/c$ as infinity which is certainly on L and hence this description also covers the linear transformations.

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Q-2) Let T be the Mobius transformation with $T(i\sqrt{3}) = \infty$, $T(-i\sqrt{3}) = 0$, and $T(0) = -1$. Find the image under T of the region R which is the intersection of the discs $|z+1| \leq 2$ and $|z-1| \leq 2$.

Solution:

First of all note that T is uniquely defined with the given data, and

$$T(z) = \frac{z + i\sqrt{3}}{z - i\sqrt{3}}.$$

The points $A = i\sqrt{3}$ and $B = -i\sqrt{3}$ are the intersection points of the circles $|z+1| = 2$ and $|z-1| = 2$. The point A is send to infinity so the two circles are send to the two lines which intersect at 0 with the same angle as the circles intersect at B.

The circle $|z-1| = 2$ is sent to the line $y = \sqrt{3}x$, and the circle $|z+1| = 2$ is sent to the line $y = -\sqrt{3}x$. The region R is sent to one of the four regions defined by these lines. Since $T(0) = -1$, the region R is sent to the region that contains -1.

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Q-3) Let R be a connected, simply connected, non-empty, proper open subset of $\mathbb{C}$. Fix a point $z_0 \in R$. Let $\mathcal{F}$ be the set of all analytic injective functions f from R into the unit disc U with $f(z_0) = 0$. Assume that $\mathcal{F}$ is not empty. Take an $f \in \mathcal{F}$. Show that if f is not surjective, then there is some $g \in \mathcal{F}$ such that $|g'(z_0)| > |f'(z_0)|$.

Solution:

Since f is not surjective, it misses a point $w \in U$. For any point $p \in U$ define

$$T_p(z) = \frac{z - p}{1 - \bar{p}z}.$$

Define a branch of logarithm, and hence a branch of the square-root function on the simply connected region $T_w(f(R))$. Let $w' = \sqrt{-w}$. Define a function $g : R \to U$ as

$$g(z) = T_{w'}(\sqrt{T_w(f(z))}).$$

We then have

$$f(z) = T_w^{-1}((T_{w'}^{-1}(g(z)))^2).$$

Notice that the map

$$s(z) = T_w^{-1}((T_{w'}^{-1}(z))^2)$$

is an automorphism of the unit disc fixing zero,and it is not an isomorphism because of the square function involved. Therefore by Schwarz Lemma (Lemma 7.2 in our textbook), we must have $|s'(0)| < 1$.

Finally, since $f = s \circ g$, using chain rule we get

$$|f'(z_0)| = |s'(0)||g'(z_0)| < |g'(z_0)|.$$

Check that $g \in \mathcal{F}$. This completes the proof.

(This is taken from: http://people.reed.edu/ jerry/311/rmt.pdf)

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Q-4) Suppose f is an entire function and there is some $R > 0$ and $z_0 \in \mathbb{C}$ such that the open ball $B_R(z_0)$ with radius R centered at z_0 is not in the range of f, i.e. $B_R(z_0) \cap f(\mathbb{C}) = \emptyset$. Show that f is constant. (In fact if the complement of $f(\mathbb{C})$ contains at least two distinct points, let alone a whole disc, then f is constant. That is Picard's theorem but you must demonstrate a proof for this homework problem only using Liouville's theorem.)

Solution:

The function $g(z) = \dfrac{1}{f(z) - z_0}$ is entire and $|g(z)| = \dfrac{1}{|f(z) - z_0|} < \dfrac{1}{R}$, so g is constant. Hence f must be constant too.