Date: 30 October 2013, Wednesday NAME:...
Time: 10:30-12:30
Ali Sinan Sertöz STUDENT NO:...

Math 302 Complex Analysis II – Midterm Exam 1 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. **A correct answer without proper reasoning may not get any credit.**

In this exam you are allowed to use two A4 size cheat-sheets provided that they are written by yourself, no photocopies are allowed. Your name must be written on both of them during the exam. **You are asked to hand in your cheat-sheets together with your answers.**

Q-1) Find the sum $\displaystyle\sum_{n=0}^{\infty}\binom{3n}{2n}\left(\frac{12}{125}\right)^n$.

Hint: You may need to know that the roots of $125z^2-12(1+z)^3=0$ *are approximately* $-\frac{1}{5}$, $\frac{2}{3}$ *and* 7.

Solution:

$$\begin{aligned}
\sum_{n=0}^{\infty}\binom{3n}{2n}\left(\frac{12}{125}\right) &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{C_R}\frac{(1+z)^{3n}}{z^{2n}}\frac{12^n}{125^n}\frac{dz}{z}\\
&= \frac{1}{2\pi i}\int_{C_1}\sum_{n=0}^{\infty}\left[\frac{12(1+z)^n}{125z^2}\right]^n\frac{dz}{z}\\
&= \frac{1}{2\pi i}\int_{C_1}\frac{125z}{125z^2-12(1+z)^3}\,dz\\
&= \text{Res}\left(\frac{125z}{125z^2-12(1+z)^3}, z=-1/5\right)+\text{Res}\left(\frac{125z}{125z^2-12(1+z)^3}, z=2/3\right)\\
&= \left(\left.\frac{125z}{250z-36(1+z)^2}\right|_{z=-1/5}\right)+\left(\left.\frac{125z}{250z-36(1+z)^2}\right|_{z=2/3}\right)\\
&= \frac{5815}{3652}\approx 1.59.
\end{aligned}$$

The actual value, calculated using actual roots, is 1.60.

NAME: STUDENT NO:

Q-2) For any integer $n > 0$, define

$$I_n = \int_{2013-i\infty}^{2013+i\infty} \frac{2014^z}{z^{n+1}}\, dz.$$

Calculate I_n.

Solution:

$$\begin{aligned} I_n &= 2\pi i \operatorname{Res}\left(\frac{2014^z}{z^{n+1}}, z=0\right) \\ &= 2\pi i \frac{f^{(n)}(0)}{n!}, \text{ where } f(z) = 2014^z, \\ &= \frac{2\pi i}{n!}\, (\ln 2014)^n. \end{aligned}$$

Here we used the principle branch in calculating 2014^z. A general branch would be

$$2014^z = \exp(z[\ln 2014 + 2\pi i k]), \text{ where } k \in \mathbb{Z}.$$

Then the answer would be

$$I_n = \frac{2\pi i}{n!}\, (\ln 2014 + 2\pi i k)^n,$$

where k is an integer and $k = 0$ represents the principle branch.

NAME: STUDENT NO:

Q-3)

(a) Calculate the cross-ratio $(i, 2i, 3, 4)$.

(b) Prove that the cross-ratio (z_1, z_2, z_3, z_4) is real if and only if the four points z_1, z_2, z_3, z_4 lie on a circle. (Here *circle* means a circle on the Riemann sphere so lines in the plane are also included in the meaning.)

Solution:

Recall that $(z_1, z_2, z_3, z_4) = \frac{z_4 - z_2}{z_4 - z_1}\frac{z_3 - z_1}{z_3 - z_2}$. Hence

$$(i, 2i, 3, 4) = \frac{4 - 2i}{4 - i}\frac{3 - i}{3 - 2i} = \frac{210}{221} + \frac{10}{221}i.$$

The transformation $T(z) = \frac{z - z_2}{z - z_1}\frac{z_3 - z_1}{z_3 - z_2}$ sends z_1, z_2, z_3 to $\infty, 0, 1$ along the real line. We also know that Mobius transformations send circles to circles. Moreover $T(z)$ lies on the same *circle* as $T(z_1), T(z_2), T(z_3)$ if and only if $T(z)$ is real, obviously! Hence z lies on the circle formed by z_1, z_2, z_3 if and only if $T(z)$ is real.

NAME: STUDENT NO:

Q-4)

(a) Does there exist an automorphism f of the unit disk $\mathbb{D}$ with the properties $f(0) = 0$ and $f'(0) = -3$?

(b) Let f_1 and f_2 be two conformal mappings of the upper half plane $\mathbb{H}$ onto the unit disk $\mathbb{D}$. Assume that for some point $z_0 \in \mathbb{H}$, we have $f_1(z_0) = f_2(z_0) = 0$ and $f_1'(z_0) > 0,\ f_2'(z_0) > 0$. Prove or disprove that $f_1 \equiv f_2$.

Solution:

If $f(0) = 0$, then $f(z) = e^{i\theta} z$ for some real θ. Since $f'(0) = e^{i\theta} = -3$ has a no solution in real θ, such functions do not exist.

For the second part let $f(z) = f_2 \circ f_1^{-1}(z)$. Then f is an automorphism of the unit disk with $f(0) = 0$ and $f'(0) = 1$. When $f(0) = 0$, we must have $f(z) = e^{i\theta} z$. Then $f'(0) = e^{i\theta} > 0$ forces $f(z) = z$. Hence $f_1 = f_2$ on $\mathbb{H}$. Observe that this argument works for any proper, non-empty, simply connected region U, and not just for $\mathbb{H}$.

NAME: STUDENT NO:

Q-5)

(a) Let U be a proper, non-empty, simply connected region U in $\mathbb{C}$. Notice that the Riemann Mapping Theorem makes no claim about the conformal equivalence of U with $\mathbb{C}$. Complete this gap by proving or disproving that there exists a region U as above which is conformally equivalent to $\mathbb{C}$.

(b) Let U be as above and let z_1, z_2 be two distinct points in U. Prove or disprove that there exists a conformal map $f : U \to U$ such that $f(z_1) = z_2$.

Solution:

Assume that there exists a conformal isomorphism $f_1 : \mathbb{C} \to U$. By the Riemann Mapping Theorem there exists a conformal map $f_2 : U \to \mathbb{D}$, where $\mathbb{D}$ is the unit disk. Then $f = f_2 \circ f_1$ is a conformal isomorphism of $\mathbb{C}$ onto the unit disk. But then for any $z \in \mathbb{C}$, we must have $|f(z)| < 1$ and hence by Liouville's theorem f is constant and cannot be onto $\mathbb{D}$. This contradiction shows that no such U exists.

For the second part let h be a conformal mapping of U onto the upper half plane $\mathbb{H}$, whose existence is guaranteed by the Riemann Mapping Theorem. Define $f : U \to U$ as $f(z) = h^{-1}(h(z) - h(z_1) + h(z_2))$. Then clearly f is a conformal map and $f(z_1) = z_2$. Here we used the fact that $\mathbb{H}$ is closed under addition which is not something we can say about an arbitrary region U.