Date:
Time:
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 302 Complex Analysis II – Makeup Exam – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

You may use the following formulas directly if you find them correct and meaningful.

The roots of $f(z) = (1+z)^3 - 9z = 0$ are

$$z_1 = 5.411474128,\ z_2 = -0.2266815976,\ z_3 = 0.8152074694.$$

Moreover

$$\frac{1}{f'(z_1)} = 0.03858848393,\ \frac{1}{f'(z_2)} = 0.1702320985,\ \frac{1}{f'(z_3)} = -0.2088205819.$$

and

$$\frac{z_1}{f'(z_1)} = 0.2088205823,\ \frac{z_2}{f'(z_2)} = -0.03858848375,\ \frac{z_3}{f'(z_3)} = -0.1702320982.$$

$$\cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \frac{2}{945}z^5 - \frac{1}{4725}z^7 - \frac{2}{93555}z^9 + \cdots$$

$$\sum_{\substack{n=-\infty \\ n\neq z_k}}^{\infty} f(n) = -\sum_k \operatorname{Res}(\pi f(z)\cot \pi z; z_k).$$

NAME: STUDENT NO:

Q-1) Find the sum $\displaystyle\sum_{n=0}^{\infty}\binom{3n}{2n}\frac{1}{9^n}$.

Solution:

$\displaystyle\binom{3n}{2n}$ is the coefficient of z^{2n} in $(1+z)^{3n}$.

When $|z|<1$, we have $\displaystyle\left|\frac{(1+z)^3}{9z^2}\right| < \frac{8}{9} < 1$, so we can write

$$\begin{aligned}
\sum_{n=0}^{\infty}\binom{3n}{2n}\frac{1}{9^n} &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{|z|=1}\left(\frac{(1+z)^3}{9z^2}\right)^n\frac{dz}{z} \\
&= \frac{1}{2\pi i}\int_{|z|=1}\sum_{n=0}^{\infty}\left(\frac{(1+z)^3}{9z^2}\right)^n\frac{dz}{z} \\
&= \frac{9}{2\pi i}\int_{|z|=1}\frac{z\,dz}{9z-(1+z)^3} \\
&= 9\times\left(\mathrm{Res}\left(\frac{z}{-f(z)}; z=z_2\right)+\mathrm{Res}\left(\frac{z}{-f(z)}; z=z_3\right)\right), \quad \text{where } f(z)=(1+z)^3-9z, \\
&= 9\times\left(\frac{z_2}{-f'(z_2)}+\frac{3}{-f'(z_3)}\right) \\
&= 9\times(0.03858848375+0.1702320982), \quad \text{(see cover page)} \\
&= 9\times 0.20882058195, \\
&= 1.87938523754,
\end{aligned}$$

since z_2 z_3 are the only poles inside the unit circle, and each is a simple pole.

NAME: STUDENT NO:

Q-2) Find the sum $\sum_{n=1}^{\infty} \frac{1}{n^6}$.

Solution:

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^6} &= 1 + \frac{1}{64} - \frac{1}{729} + \cdots \\
&= \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n^6} \\
&= -\frac{1}{2} \operatorname{Res}(\frac{\pi}{z^6} \cot \pi z; 0) \\
&= -\frac{\pi}{2} \left(-\frac{2\pi^5}{945} \right) \\
&= \frac{\pi^6}{945} \approx 1.017343062.
\end{aligned}
$$

NAME: STUDENT NO:

Q-3) Evaluate the integral $\displaystyle\int_{|z|=1/2} \frac{\cot(z)}{z^4 - z^5}\, dz$.

Solution:

The only singularity in the given circle is $z = 0$. Here we also have

$$\begin{aligned}
\frac{\cot(z)}{z^4 - z^5} &= \frac{1}{z^4}\frac{\cot z}{1 - z} \\
&= \frac{1}{z^4}[\cot(z) \cdot (1 + z + z^2 + \cdots + z^n + \cdots)] \\
&= \frac{1}{z^4}[\frac{1}{z} + 1 + \frac{2}{3}z + \frac{2}{3}z^2 + \frac{29}{45}z^3 + \frac{29}{45}z^4 + \frac{607}{945}z^5 + \cdots] \\
&= \cdots + \frac{29}{45z} + \cdots
\end{aligned}$$

Hence the value of the integral is $2\pi i\,\dfrac{29}{45} = \dfrac{58\pi}{45}\, i \approx 4.049163866\, i$.

NAME: STUDENT NO:

Q-4) Let $A_n = \{a_1, \ldots, a_n\}$ and $B_m = \{b_1, \ldots, b_m\}$ be two subsets of $\mathbb{C}$ of cardinalities n and m respectively, for $n, m > 0$. Set $A_0 = B_0 = \emptyset$.

(a) Suppose that there is a Mobius transformation ϕ such that $\phi : \mathbb{C} \setminus A_n \to \mathbb{C} \setminus B_m$ is an isomorphism. Show that $m = n$.

(b) Suppose that $n = m$. Can we find a Mobius transformation ϕ which sends $\mathbb{C} \setminus A_n$ isomorphically onto $\mathbb{C} \setminus B_n$?

Solution:

Part (a):
We have two cases:
Case 1: $\phi(\infty) = \infty$.
In this case we must have $\phi(A_n) = B_m$, and since ϕ is injective we must have $n = m$.

Case 2: $\phi(\infty) \in \mathbb{C}$.
Since ϕ is injective, no point of $\mathbb{C}$ will be mapped to $\phi(\infty)$, so $\phi(\infty) \in B_m$. Say $\phi(\infty) = b_m$. Since ϕ is onto $\mathbb{C} \cup \{\infty\}$, there must be a point in $\mathbb{C}$ which maps to ∞. Since $\mathbb{C} \setminus A_n$ is mapped into $\mathbb{C}$, this point must be in A_n. Say $\phi(a_n) = \infty$. We now have $\phi(A_n \setminus \{ a_n\}) = B_m \setminus \{b_m\}$. Since ϕ is injective, we must have $n = m$.

Part (b):
Case 1: $n = 0$. In this case $\phi(z) = z$ is the required isomorphism.

Case 2: $n = 1$. In this case $\phi(z) = z - a_1 + b_1$ is the required isomorphism.

Case 3: $n = 2$. In this case $\phi(z) = \left(\dfrac{b_1 - b_2}{a_1 - a_2}\right) z + \left(\dfrac{a_1 b_2 - a_2 b_1}{a_1 - a_2}\right)$ is the required isomorphism.

Case 4: $n \geq 3$.
If we require that $\phi(\infty) = \infty$, then ϕ must be linear and is totally determined by a_1, a_2, b_1, b_2 as in Case 3 above. Then $\phi : \mathbb{C} \setminus A_n \to \mathbb{C} \setminus B_n$ is an isomorphism if and only if $b_j = \phi(a_j)$ for $j = 3, \ldots, n$. So the existence depends not only on the cardinality of B_n but also on its elements.

If on the other hand we require that $\phi(\infty) \in \mathbb{C}$, then arguing as we did above in Case 2 of Part (a), we will have $\phi(\infty) = b_n$ and $\phi(a_n) = \infty$. Then we are asking if $\phi(A_n \setminus \{a_n\}) = B_n \setminus \{b_n\}$. But a Mobius transformation is determined by its action on three points. We already specified its value on two points so we have margin for one more arbitrary assignment. We can set $\phi(a_1) = b_1$. Then $\phi : \mathbb{C} \setminus A_n \to \mathbb{C} \setminus B_n$ is an isomorphism if and only if $b_j = \phi(a_j)$ for $j = 2, \ldots, n-1$. When $n \geq 3$, this already gives at least one more condition so in general it will not be satisfied. Again we conclude that the existence of an isomorphism as asked in the question depends not only on the cardinality of B_n but also on its elements.