Date: 10 March 2016, Thursday
Time: 08:40-10:30
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:......................................

Math 302 Complex Analysis II – Midterm 1 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

You may use the following formulas directly if you find them correct and meaningful.

$$\cot z = \frac{1}{z} - \frac{1}{3}\,z - \frac{1}{45}\,z^3 - \frac{2}{945}\,z^5 - \frac{1}{4725}\,z^7 - \frac{2}{93555}\,z^9 + \cdots$$

$$\csc z = \frac{1}{z} + \frac{1}{6}\,z + \frac{7}{360}\,z^3 + \frac{31}{15120}\,z^5 + \frac{127}{604800}\,z^7 + \frac{73}{3421440}\,z^9 + \cdots$$

$$\sum_{\substack{n=-\infty \\ n\neq z_k}}^{\infty} f(n) = -\sum_k \operatorname{Res}(\pi f(z)\cot \pi z; z_k), \qquad \sum_{\substack{n=-\infty \\ n\neq z_k}}^{\infty} (-1)^n f(n) = -\sum_k \operatorname{Res}(\pi f(z)\csc \pi z; z_k)$$

$$\binom{n}{k} = \frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^k}\,\frac{dz}{z}$$

The roots of $f(z) = z^3 + 3z^2 - 6z + 1$ are $z_1 = 1.22$, $z_2 = -4.41$, $z_3 = 0.18$. Moreover

$$\frac{1}{f'(z_1)} = 0.17, \quad \frac{1}{f'(z_2)} = 0.03, \quad \frac{1}{f'(z_3)} = -0.20.$$

NAME: STUDENT NO:

Q-1) Let R be a non-empty simply connected open subset of $\mathbb{C}$, and let z_1, z_2 be two distinct points in R.

1. Show that there exists a conformal mapping of R onto itself, taking z_1 into z_2, when $R \subsetneq \mathbb{C}$.
2. Show that there exists a conformal mapping of R onto itself, taking z_1 into z_2, when $R = \mathbb{C}$.
3. Show that there exists no conformal mapping of R onto $\mathbb{C}$ when $R \subsetneq \mathbb{C}$.

Solution:

When $R \subsetneq \mathbb{C}$, let ϕ and ψ be two conformal maps of R onto the unit disk with $\phi(z_1) = 0$ and $\Psi(z_2) = 0$. The existence of such maps is given by Riemann Mapping Theorem. Then $\psi^{-1} \circ \phi$ is a conformal map of R onto itself sending z_1 into z_2.

When $R = \mathbb{C}$, the linear map $z \mapsto (z - z_1) + z_2$ is a conformal map of R onto itself sending z_1 into z_2.

When $R \subsetneq \mathbb{C}$, assume that there is a conformal map ϕ of R onto $\mathbb{C}$. Let ψ be a conformal map of R onto the unit disk. Then $\psi \circ \phi^{-1}$ is an entire function whose modulus is bounded by 1. Then it must be constant by Louville's theorem, but this is contrary to the way it is constructed as composition of two conformal maps. This contradiction shows that there is no such conformal map of R onto $\mathbb{C}$.

NAME: STUDENT NO:

Q-2) Find the sum $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$.

Solution:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} &= -1 + \frac{1}{16} - \frac{1}{81} + \cdots \\
&= \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n^4} \\
&= -\frac{1}{2} \operatorname{Res}\left(\frac{\pi}{z^4} \csc \pi z; 0\right) \\
&= -\frac{\pi}{2} \, \frac{7\pi^3}{360} \\
&= -\frac{7\pi^4}{720} \approx -0.947.
\end{aligned}$$

NAME: STUDENT NO:

Q-3) Find the sum $\sum_{n=0}^{\infty} \binom{3n}{n} \frac{1}{9^n}$.

Solution:

When $|z| < 1$, we have $\left| \frac{(1+z)^3}{9z} \right| < \frac{8}{9} < 1$, so we can write

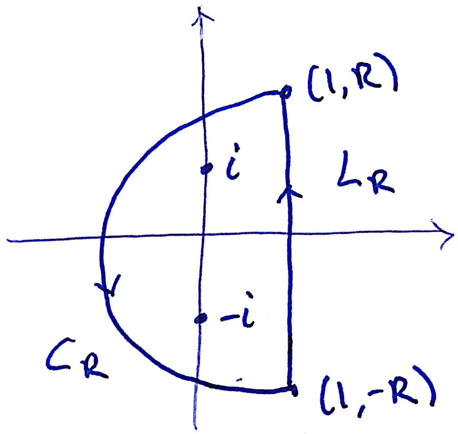
$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{3n}{n} \frac{1}{9^n} &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{|z|=1} \left(\frac{(1+z)^3}{9z} \right)^n \frac{dz}{z} \\
&= \frac{1}{2\pi i} \int_{|z|=1} \sum_{n=0}^{\infty} \left(\frac{(1+z)^3}{9z} \right)^n \frac{dz}{z} \\
&= \frac{9}{2\pi i} \int_{|z|=1} \frac{dz}{9z - (1+z)^3} \\
&= 9 \times \mathrm{Res}(\frac{1}{-f(z)}; z = z_3), \quad \text{where } f(z) = (1+z)^3 - 9z, \\
&= 9 \times \frac{1}{-f'(z_3)} \\
&= 9 \times (0.20), \quad \text{(see cover page)} \\
&= 1.80,
\end{aligned}$$

since z_3 is the only pole inside the unit circle, and it is a simple pole. If we used more digits in the approximation of the roots of $f(z)$, we would be able to get a more accurate estimate for the sum whic is around 1.87.

NAME: STUDENT NO:

Q-4) Evaluate the integral $\frac{1}{2\pi i}\int_L \frac{e^z}{1+z^2}\,dz$, where L is the line $1+it$ for $t\in\mathbb{R}$, directed upwards.

Solution:



Let L_R be the line parametrized by $z=1+it$ for $t\in[-R,R]$ where $R>\sqrt{2}$. Also let C_R be the semicircle parametrized by $z=1+Re^{i\theta}$ where $\theta\in[2\pi/2,3\pi/2]$. Let $\gamma_R=L_R+C_R$. Then we have

$$\frac{1}{2\pi i}\int_{\gamma_R}\frac{e^z}{1+z^2}\,dz=\mathrm{Res}(\frac{e^z}{1+z^2}\,dz,z=i)+\mathrm{Res}(\frac{e^z}{1+z^2}\,dz,z=-i)=\sin 1=.8414709848....$$

We also have

$$\frac{1}{2\pi i}\int_{\gamma_R}\frac{e^z}{1+z^2}\,dz=\frac{1}{2\pi i}\int_{C_R}\frac{e^z}{1+z^2}\,dz+\frac{1}{2\pi i}\int_{L_R}\frac{e^z}{1+z^2}\,dz.$$

For $z\in C_R$ we have

$$\left|\frac{e^z}{1+z^2}\right|\le\frac{e^{1+R\cos\theta}}{R^2-1}\le\frac{e}{R^2-1},$$

since $\cos\theta<0$ on C_R. Hence we have

$$\left|\frac{1}{2\pi i}\int_{C_R}\frac{e^z}{1+z^2}\,dz\right|\le\frac{Re}{R^2-1}\to 0\quad\text{as } R\to\infty\quad.$$

Putting these together we find

$$\frac{1}{2\pi i}\int_L\frac{e^z}{1+z^2}\,dz=\sin 1=.8414709848....$$