Bilkent University

Exam # 01
Math 430 Introduction to Complex Geometry
Due: 12 March 2020
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**Q-1)** Let $F_n = \{x + iy \in \mathbb{C} \mid -\infty < x < \infty, \;\; (2n-1)\pi < y < (2n+1)\pi \}$, where $n \in \mathbb{Z}$. Also let $G = \mathbb{C} \setminus \{x + iy \in \mathbb{C} \mid y = 0, \;\; x \leq 0 \}$, where $\setminus$ denotes subtraction of sets.
Define a map $\exp : \mathbb{C} \to \mathbb{C}$ by $\exp(z) = e^x \cos y + i e^x \sin y$.

(i) Show that $\exp$ is analytic.

(ii) Show that $\exp$ is the unique analytic extension to $\mathbb{C}$ of the real analytic function $e^x$.

(iii) Show that $\exp : F_n \to G$ is one-to-one and onto.

(iv) Define an inverse of $\exp : F_n \to G$ for each $n$. It is easier to describe the inverse function using polar coordinates in $G$.

(v) Show that each of the above inverses, which we call a branch of the log function, is analytic. It is again easier here to use the polar version of the Cauchy-Riemann equations.

**Q-2)** Let $f(w) = f(s,t)$ where $w = s + it$, and $g(z) = g(x,y)$ where $z = x + iy$ be two $C^\infty$ functions of the real variables $s, t$ and $x, y$ respectively. Assume that $\phi(z) = (f \circ g)(z)$ is defined. Show that

$$\frac{\partial \phi}{\partial z} = \frac{\partial f}{\partial w}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{g}}{\partial z},$$

and

$$\frac{\partial \phi}{\partial \bar{z}} = \frac{\partial f}{\partial w}\frac{\partial g}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{g}}{\partial \bar{z}},$$

where the derivatives of $f$ are evaluated at $w = g(z)$.

**Q-3)** Let $f(w_1, \ldots, w_n)$ be a $C^\infty$ function of the real variables $s_1, t_1, \ldots, s_n, t_n$ where each $w_k = s_k + it_k$. Assume further that each $w_k = g_k(z_1, \ldots, z_m)$ is a $C^\infty$ function of the real variables $x_1, y_1, \ldots, x_m, y_m$, where each $z_j = x_j + iy_j$.
Using the previous result, give a convincing argument that for each $j = 1, \ldots, m$,

$$\frac{\partial f \circ g}{\partial z_j} = \sum_{k=1}^{n} \frac{\partial f}{\partial w_k}\frac{\partial g_k}{\partial z_j} + \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{w}_k}\frac{\partial \overline{g}_k}{\partial z_j},$$

and

$$\frac{\partial f \circ g}{\partial \overline{z}_j} = \sum_{k=1}^{n} \frac{\partial f}{\partial w_k}\frac{\partial g_k}{\partial \overline{z}_j} + \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{w}_k}\frac{\partial \overline{g}_k}{\partial \overline{z}_j},$$

where each derivative of $f$ is evaluated at $(g_1(z), \ldots, g_n(z))$.