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Math 431 ALGEBRAIC GEOMETRY
Final Exam
Solutions

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1) Let f and g be meromorphic functions on a compact Riemann surface C such that $(f)_\infty = \sum_{i=1}^{N} n_i p_i$ and $(g)_\infty = \sum_{i=1}^{N} m_i p_i$ where $n_i, m_i \geq 0$. Show that there exists $\alpha \in \mathbb{C}$ such that $(f+\alpha g)_\infty = \sum_{i=1}^{N} \max\{n_i, m_i\} p_i$. Now let $D = \sum_{i=1}^{N} k_i p_i$ be an effective divisor on C with the property that for every $p \in C$, $\mathfrak{L}(D-p) \subsetneq \mathfrak{L}(D)$. Show that there exists a meromorphic function h on C with $(h)_\infty = D$.

Fix an i and let $n = n_i$, $m = m_i$ and $p = p_i$. Without loss of generality assume that $n \geq m$. Let z be a local coordinate centered at p. The principal parts of f and g at p are

$$\frac{a_n}{z^n} + \cdots + \frac{a_1}{z} \quad \text{and} \quad \frac{b_m}{z^m} + \cdots + \frac{b_1}{z}$$

respectively, where $a_n b_m \neq 0$. Now observe that if $n > m$, then for every $\alpha \in \mathbb{C}$, the function $f + \alpha g$ has a pole of order n at p. If however $n = m$, then for all $\alpha \in \mathbb{C}$, except for $\alpha = -a_n/b_m$, the function $f + \alpha g$ has a pole of order n at p. Repeating this for each i we see that for all $\alpha \in \mathbb{C}$, except for finitely many values, the function $f + \alpha g$ has the required pole properties.

Now for the second part first observe that for every $f \in \mathfrak{L}(D)$, the order of the pole of f at p_i is at most k_i. Using the given property of D we can choose for each i, a function $f_i \in \mathfrak{L}(D) \backslash \mathfrak{L}(D-p_i)$, for which the order of pole at p_i is exactly k_i. Using the first part we now know that there exist constants $\alpha_2, \ldots, \alpha_N$ such that the function $h = f_1 + \alpha_2 f_2 + \ldots + \alpha_N f_N$ has the required pole properties.

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2) If C is a compact Riemann surface of genus $g \geq 1$, then show that for every pair of distinct points $p, q \in C$, there exists a meromorphic 1-form ω such that $(\omega)_\infty = p+q$. Moreover show that for any nonzero constant $\alpha \in \mathbb{C}$, we can further choose this ω such that $\text{Res}_p(\omega) = \alpha$.

Using Riemann-Roch theorem we find that $i(-p-q) = g+1$, i.e. there are $g+1$ meromorphic differential forms ω with $(\omega) \geq -p-q$, or equivalently $(\omega)_\infty \leq p+q$. We know that there are exactly g holomorphic forms, for which we have $(\omega)_\infty = 0$. Since $i(-p-q) = g+1$, there is one meromorphic form for which we have $0 < (\omega)_\infty \leq p+q$. Suppose $(\omega)_\infty = p$. Then ω has only one simple pole at p with a necessarily nonzero residue, which contradicts with the fact that the sum of residues of a form must be zero. So we must have $(\omega)_\infty = p+q$ as required. If $\text{Res}_p(\omega) = c$, where c is necessarily nonzero, then the form $(\alpha/c)\omega$ is the form asked in the question.

This is theorem 2.3 on page 161 where it is proved using projective techniques. After the proof there is a note saying that the theorem follows immediately from Riemann-Roch.

3) Let C be a compact Riemann surface of genus g, and D a divisor on C of degree d with $d > 2g$. Show that $\ell(D) = d - g + 1$. Let $f_0, \dots, f_{d-g}$ be a basis of $\mathfrak{L}(D)$. Let $\phi : C \to \mathbb{P}^{d-g}$ be defined by $\phi(p) = [f_0(p) : \cdots : f_{d-g}(p)]$. Show that ϕ is well defined. Show that ϕ is injective.

That $\ell(D) = d - g + 1$ follows immediately from Riemann-Roch theorem after observing that $\deg(K - D) < -2$. In fact the same observation implies, using again the Riemann-Roch theorem, that $\mathfrak{L}(D - p - q) \subsetneq \mathfrak{L}(D - p) \subsetneq \mathfrak{L}(D)$.

ϕ would be undefined at a point p if each f_i vanishes at p, but this would contradict $\mathfrak{L}(D-p) \subsetneq \mathfrak{L}(D)$.

Now suppose there exist $p \neq q$ in C with $\phi(p) = \phi(q)$. Write D as $D = np + mq + D'$ where p and q does not appear in D'. Here n and m are integers, possibly zero. Let z_1 and z_2 be local coordinates centered at p and q respectively. We then have $f_i(z_1) = z_1^n g_i(z_1)$ and $f_i(z_2) = z_2^m h_i(z_z)$, $i = 0, \dots, d - g$, where g_i and h_i are holomorphic. Then we have

$$\phi(p) = [g_0(0) : \cdots : g_{d-g}(0)] \text{ and } \phi(q) = [h_0(0) : \cdots : h_{d-g}(0)].$$

Since $\phi(p) = \phi(q)$, there is a nonzero $\lambda \in \mathbb{C}$ such that $g_i(0) = \lambda h_i(0)$, $i = 0, \dots, d - g$. Let $f \in \mathfrak{L}(D-p) \backslash \mathfrak{L}(D-p-q)$. We have $f(z_1) = \sum c_i f_i(z_1) = z_1^n \sum c_i g_i(z_1)$, and since $f \in \mathfrak{L}(D-p)$, we must have $\sum c_i g_i(0) = 0$. On the other hand $f(z_2) = \sum c_i f_i(z_2) = z_2^m \sum c_i h_i(z_2)$, and since $f \notin \mathfrak{L}(D - p - q)$, we must have $\sum c_i h_i(0) \neq 0$. We now have a contradiction;

$$0 = \sum c_i g_i(0) = \lambda \sum c_i h_i(0) \neq 0.$$

Thus ϕ is injective.

Short proof: $\phi(p) = \phi(q) \implies f_i(p) = \lambda f_i(q)$. Since $f = \sum c_i f_i$, $f(p) = 0 \iff f(q) = 0$, and this contradicts $\mathfrak{L}(D - p) \subsetneq \mathfrak{L}(D)$.

This is an exercise on page 124 which I solved in class.

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4) Describe the canonical map of a compact Riemann surface C of genus $g \geq 2$, and show that the canonical map is nondegenerate, i.e. the image does not lie in any hyperplane.

Let $\omega_1, \dots, \omega_g$ be a basis of $\Omega^1(C)$. The canonical map ϕ_K is the map from C to $\mathbb{P}^{g-1}$ defined by $\phi_K(p) = [\omega_1(p) : \cdots : \omega_g(p)]$. If the map is degenerate then for all $p \in C$ the image lies in a hyperplane, i.e. $\sum \lambda_i \omega_i(p) = 0$ for all p. But this gives $\sum \lambda_i \omega_i = 0$ contradicting the fact that the ω_i's form a basis.

The definition is on page 133, the proof is on page 134.

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5) Let C be a compact Riemann surface, K a canonical divisor and D any divisor. Show that $i(D) = \ell(K - D)$.

Let ω_0 be a differential form with $(\omega_0) = K$. Consider the map from $\mathfrak{L}(K - D)$ to $K^1(D)$ given by $f \mapsto f\omega_0$. The inverse is given by $\omega \mapsto \omega/\omega_0$. Hence the two vector spaces are isomorphic and their dimensions are the same.

This is a special case of the theorem on page 101.
