Date: May 23, 2008, Friday NAME:..
Time: 12:15-14:15, SAZ04
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Math 431 Algebraic Geometry – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. A correct answer without proper reasoning may not get any credit.

All curves today are smooth projective curves. The following are some formula without any further details. Use them at your own jurisdiction.

$$\ell(D) - \ell(K - D) = \deg D - g + 1.$$

If $f : X \to Y$ is of degree d, then

$$2g_X - 2 = d(2g_Y - 2) + \deg R, \quad \text{where} \quad R = \sum_{p \in X} (e_p - 1) \cdot p.$$

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

$$D \sim 0 \iff D = (f).$$

NAME: STUDENT NO:

Q-1) Show that in $\mathbb{P}^1$ any two points p and q are linearly equivalent, i.e. there exists a rational function f such that $(f) = p - q$. Show that the genus of $\mathbb{P}^1$ is zero and that it is not possible to have an un-ramified morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree greater than 1.

Solution:

Let $p = [a_0 : a_1]$ and $q = [b_0 : b_1]$ in $\mathbb{P}^1$. That $p \neq q$ means $a_0b_1 - a_1b_0 \neq 0$. Let $[x_0 : x_1]$ be homogenous coordinates on $\mathbb{P}^1$.

Define $f(x_0, x_1) = \dfrac{a_1x_0 - a_0x_1}{b_1x_0 - b_0x_1}$. Since $p \neq q$, f is not constant. Then it is clear that $(f) = p - q$.

The above calculation shows that $f \in L(q)$. Clearly constants are also in $L(q)$, so $\ell(q) \geq 2$. In particular for all positive integers n, we see that the functions $1, f, f^2, \ldots, f^n$ are all in the vector space $L(nq)$ and hence $\ell(nq) \geq n + 1$. Let $n > degK$ where K is the canonical divisor of $\mathbb{P}^1$. Then $\ell(K - nq) = 0$ and the Riemann-Roch theorem for the divisor nq becomes

$$g = \deg nq + 1 - \ell(nq) \leq 0,$$

but since the genus is always a non-negative integer, we get that the genus of $\mathbb{P}^1$ is zero.

NAME: STUDENT NO:

Q-2) Show that every curve of genus 1 can be embedded into $\mathbb{P}^2$ as a curve of degree 3 of the form $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in \mathbf{k}$.

Solution:

Let C be a curve of genus 1 and let K be its canonical divisor. $\deg K = 2g - 2 = 0$ so for any divisor D of positive degree we have $\ell(K - D) = 0$. The Riemann-Roch theorem for such D becomes

$$\ell(D) = \deg D.$$

In particular, fix a point $p \in C$ and let n be any positive integer. Then $\ell(np) = n$. If x and y are rational functions on C with the property of having a double (resp triple) pole at p and being regular elsewhere, then:
$L(p)$ is generated by 1.
$L(2p)$ is generated by $\{1, x\}$.
$L(3p)$ is generated by $\{1, x, y\}$.
$L(4p)$ is generated by $\{1, x, y, x^2\}$.
$L(5p)$ is generated by $\{1, x, y, x^2\, xy\}$.
We notice now that $L(6p)$ contains the functions $\{1, x, y, x^2\, xy, x^3, y^2\}$ but since $\ell(6p) = 6$, these seven functions must be linearly dependent. So there exists constants $a_1, \ldots, a_7$, not all zero, such that

$$a_1\, 1 + a_2\, x + \cdots + a_6\, x^2 + a_7\, y^3 = 0.$$

If $a_6 = a_7 = 0$, this gives linear dependence among the first five elements which we know to be linearly independent from above. On the other hand since x^3 and y^2 are the only two functions in this list with sixth order pole at p, if one of them appear in the linear dependence equation, then the other must also appear to cancel its pole. So both a_6 and a_7 are non-zero.

Now we can consider the map

$$\phi : C \to \mathbb{P}^2$$

given by $q \mapsto [1 : x(q) : y(q)]$. The above linear dependence equation shows that the image satisfies a cubic polynomial in x and y. Using Riemann-Roch theorem we easily check the criteria for this map to be an embedding (see class notes.). Then by using successive change of variables we bring the above cubic to the required form. Note that $\lambda \neq 0, 1$.

NAME: STUDENT NO:

Q-3) Find the j-invariant of the plane curve given by

$$y^2 = x^3 + 5x^2 + 11x - 2xy + 6.$$

Solution:

The above equation is equivalent to $(y+x)^2 = (x+1)(x+2)(x+3)$. Let $Y = x+y$ and $X = x+2$. Then this equation becomes $Y^2 = X(X-1)(X-\lambda)$ where $\lambda = -1$, and this gives $j = 1728$.

NAME: STUDENT NO:

Q-4) If X is a curve of genus 2 and $\phi : X \to \mathbb{P}^1$ is a morphism of degree 2, then show that ϕ is ramified at exactly 6 points each with ramification index 2.

Solution:

Using Hurwitz formula we find that $\deg R = 6$, but $\deg R = \sum_{p \in X}(e_p - 1)$ where $e_p = 1, 2$ since $\deg \phi = 2$. This gives that $e_p = 6$ for exactly six points $p \in X$.

NAME: STUDENT NO:

Q-5) Let D be a nonzero divisor of degree 0 on a curve X of genus g. Determine $\ell(K-D)$ explicitly, where K is a canonical divisor of X. Note that the answer depends on whether $D \sim 0$ or not.

Solution:

If $D \sim 0$ then the vector spaces $L(D)$ and $L(0)$ are isomorphic. But clearly $\ell(0) = 1$ since only constants are in $L(0)$. If on the other hand $D \not\sim 0$, then assume $f \in L(D)$ for some rational function f. We will have by definition of $L(D)$ that $(f) + D \geq 0$ but clearly $\deg((f) + D) = \deg((f)) + \deg(D) = 0$, so we must have $(f) + D = 0$. This shows that $D = (1/f)$, or equivalently $D \sim 0$, a contradiction. So $\ell(D) = 0$ in this case.

Now using the Riemann-Roch theorem, we find that

$$\ell(K-D) = \ell(D) + g - 1 - \deg D,$$

we find that if $D \sim 0$, then $\ell(K-D) = g$. Otherwise $\ell(K-D) = g - 1$.