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Math 114 Algebraic Geometry – Homework 2 – Solutions

Textbook: Phillip A. Griffiths, Introduction to Algebraic Curves, AMS Publications, 1989.

Q-1) Show that any two irreducible conics in $\mathbb{P}^2$ are projectively equivalent and moreover that any such conic is isomorphic to $\mathbb{P}^1$.

Solution:

Consider any homogeneous quadratic equation in the homogeneous variables x, y, z. Dehomogenize with respect to z and by change of variables, basically by completing to squares, bring it to the form $x^2 + y^2 + 1$. That the constant term is not zero is a consequence of the fact that the quadratic equation is irreducible. Now homogenize with respect to z and write

$$x^2 + y^2 + z^2 = (x - iy)(x + iy) - (iz)^2 = UV - W^2 = XZ - Y^2 = 0.$$

All the above change of coordinates are linear transformations. Hence every irreducible quadric is projectively equivalent to the quadric $XZ - Y^2$.

Now consider the map $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ given by $\phi([s : t]) = [s^2 : st : t^2]$. This map is injective. Let $W = \{[X : Y : Z] \in \mathbb{P}^2 \,|\, XZ - Y^2 = 0\}$. Clearly $\phi(\mathbb{P}^1) \subset W$ We now show that it is an isomorphism by constructing an inverse. Let $[a : b : c] \in W$. Define $\psi : W \to \mathbb{P}^2$ as follows.

$$\psi([a : b : c]) = \begin{cases} [a : b] & \text{if } a \neq 0, \\ [b : c] & \text{if } b \neq 0. \end{cases}$$

Check that ψ is well defined and actually constitutes an inverse for ϕ.

Thus all quadrics are isomorphic to $\mathbb{P}^1$.

Q-2) Let $X \subset \mathbb{P}^3$ be given as the zero set of the polynomials

$$x_0x_3 - x_1x_2 = 0,\ x_1^2 - x_0x_2 = 0,\ x_2^2 - x_1x_3 = 0.$$

And let $Y \subset \mathbb{P}^3$ be given by the parametrization

$$x_0 = s^3,\ x_1 = s^2t,\ x_2 = st^2,\ x_3 = t^3,\ [s:t] \in \mathbb{P}^1.$$

Show that X is smooth and that $X = Y$. Moreover show that no two of the defining equations for X suffice to define it.

Solution:

Let

$$f = x_0x_3 - x_1x_2 = 0,\ g = x_1^2 - x_0x_2 = 0,\ h = x_2^2 - x_1x_3 = 0.$$

Check that $x_0 = 0$ is not in the solution set. Set $x_0 = 1$. From $g = 0$, we get $x_2 = x_1^2$. And from $h = 0$ we get $x_3 = x_1^3$ when $x_1 \neq 0$. When $x_1 = 0$, the point $[1:0:0:0]$ is in the solution space. Now check that the solution space is exactly Y.

To show smoothness we consider the Jacobian matrix.

$$\begin{pmatrix} f_{x_0} & f_{x_1} & f_{x_2} & f_{x_3} \\ g_{x_0} & g_{x_1} & g_{x_2} & g_{x_3} \\ h_{x_0} & h_{x_1} & h_{x_2} & h_{x_3} \end{pmatrix} = \begin{pmatrix} x_3 & -x_2 & -x_1 & x_0 \\ -x_2 & 2x_1 & -x_0 & 0 \\ 0 & -x_3 & 2x_2 & -x_1 \end{pmatrix}$$

When $x_0 \neq 0$, then the first and second rows are linearly independent. When $x_0 = 0$, this forces $x_3 \neq 0$ and then the first and the third rows are linearly independent. Thus the rank of the Jacobian is two, matching the codimension of X. Hence X is smooth.

Clearly $Y \subset Z(f,g)$. But check that $x = 0$ forces $x_1 = 0$ and leaves x_2 and x_3 free in $Z(f,g)$. Thus $Z(f,g) \simeq Y \cup \mathbb{P}^1$. Similar arguments show that $Z(f,h)$ and $Z(g,h)$ are also isomorphic to $Y \cup \mathbb{P}^1$.

If you set $k = x_0x_3^2 - 2x_1x_2x_3 + x_2^3$, then you can show that $Y = Z(g,k)$. But you may find it interesting to check that although $k = x_3 f + x_2 h$, as homogeneous ideals in $\mathbb{C}[x_0, \ldots, x_3]$, $(g,k) \subsetneq (f,g,h)$.

Q-3) Find conditions for a and b which would ensure that $y^2 = 4x^3 + ax + b$ to be an equation representing a smooth curve.

Solution:

Let $f(x,y) = 4x^3 + ax + b - y^2$. Then $f_y = 0$ forces $y = 0$. Solving $f(x,0) = 0$ and $f_x(x,0) = 0$ together, we get $x^3 = b/8$. Putting this back into $f(x,0) = 0$ we get $27b^2 + a^3 = 0$. Thus for $f(x,y) = 0$ to represent a smooth curve, we must have $27b^2 + a^3 \neq 0$. This is the discriminant of the polynomial $f(x,0)$ and corresponds to saying that the curve $f(x,y) = 0$ is smooth if the polynomial $f(x,0)$ has no repeated roots.
