Due Date: 21 May 2012, Monday until 17:30 NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 431 Algebraic Geometry – Final and Make-up Exams – Solutions

Final Exam

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Make-up Exam

M1	M2	TOTAL
50	50	100

Please do not write anything inside the above boxes!

Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.

We work over the complex numbers.

This is an open book, open notebook Take-Home Exam.

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Q-1) On page 20 of the textbook we have Example 1.5 which gives the formula for the arc-length of an ellipse. Derive this formula and calculate, using a software if necessary, the length of the circumference of the ellipse given by

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

Solution:

Use the usual formula

$$\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

for arc-length and simplify to obtain the formula given in the book.

Maple gives 25.53 for the length of the above ellipse.

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Q-2) Let $f(x,y)$ and $g(x,y)$ be two irreducible polynomials with complex coefficients. Assume that the affine curves $Z(f)$ and $Z(g)$ never intersect. What happens at infinity? Construct two such polynomials and illustrate your proof.

Solution:

x^2+y and x^2+y+1 are two such polynomials. Their curves intersect at $[0:1:0]$ at infinity.

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Q-3) In algebraic geometric proofs we repeatedly used change of coordinates to make coordinates of points amenable to further calculations. In particular we used several times a projective transformation of $\mathbb{P}^2$ which sends a given point $[a : b : c]$ to $[1 : 0 : 0]$. Construct explicitly one such projective transformation. Can you construct a projective transformation of $\mathbb{P}^2$ which sends any three distinct points to any three distinct points in $\mathbb{P}^2$?

Solution:

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Q-4) Let $f : C \to C'$ be a non-constant algebraic map of projective smooth plane curves. Show that $g(C) \geq g(C')$, where $g(\cdot)$ denotes the genus.

Solution:

If $g(C') = 0$, there is nothing to prove. If $g(C') \geq 1$, re-write Riemann-Hurwitz formula as

$$g(C) = g(C') + (n-1)(g(C') - 1) + \frac{1}{2} \deg R,$$

where R is the ramification divisor. Since $n - 1 \geq 0$ and $g(C') - 1 \geq 0$ and $\deg R \geq 0$, we are done.

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Q-5) Let $C \subset \mathbb{P}^2$ be a non-singular algebraic curve. Let a line L intersect C at the distinct points $p_1, p_2, p_3, p_4, p_5, p_6$.
Choose a point $a \in C - \{p_1, p_2, p_3, p_4, p_5, p_6\}$.
Let L_i be the line joining a to p_i, and let D_i be the divisor consisting of the points $C \cap L_i$, counting multiplicities, $i = 1, \ldots, 6$.
Let $D = D_1 + \cdots + D_6 - 6a - p_1 - \cdots - p_6$.
Calculate $\ell(D)$.

Solution:

From Bezout's theorem we conclude that $\deg C = 6$, and from degree-genus formula we find that the genus g of C is 10. If K is a canonical divisor, it follows from $\deg K = 2g - 2$ that $\deg K = 18$.

We find that $\deg D = 24$ so that $\deg(K - D) < 0$ and hence $\ell(K - D) = 0$.

Now from Riemann-Roch theorem we find that $\ell(D) = \deg D + 1 - g = 24 + 1 - 10 = 15$.

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MQ-1) Let X be a smooth projective plane curve of genus $g > 0$, and let K be a canonical divisor of X. Show that we can choose K to be an effective divisor of the form $K = a_1 + \cdots + a_g$, where a_i are points on X. Show that $\ell(K - a_i) = g - 1$, for any $i = 1, \ldots, g$. Moreover show that $\ell(K - a_i - a_j) = g - 2$, if X is not hyperelliptic, where $1 \leq i \neq j \leq g$. Finally show that if $g = 2$, then X is hyperelliptic.

Solution:

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MQ-2) Let $f(x)$ and $g(x)$ be complex polynomials of degrees n and $n+2$ respectively, where n is a positive integer. Assume that the roots of $f(x)g(x) = 0$ are all distinct. Denote the homogenization of f and g with respect to the variable z by $f^h(x,z)$ and $g^h(x,z)$ respectively. Define a polynomial $P(x,y,z) = f^h(x,z)y^2 + g^h(x,z)$. Let X be the projective plane curve $Z(P)$. Calculate explicitly the genus of X.

Solution: