Due Date: 16 February 2012, Thursday NAME:..

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Math 431 Algebraic Geometry – Homework – Solutions

previous	5	6	TOTAL
40	10	10	60

Please do not write anything inside the above boxes!

Q-5) Find the singular points and the multiplicities and the nature (i.e. ordinary or not) of the singular points of the following projective curves.

(i) $y^2z = x(x-z)(x-\lambda z)$, $\lambda \in \mathbb{C}$.

(ii) $x^n + y^n + z^n = 0$, where $n > 0$ is an integer.

Answer:

(i) Let $f(x,y,z) = y^2z - x\,(x-z)\,(x-\lambda\,z) = y^2z - x^3 + x^2\lambda\,z + zx^2 - x\lambda\,z^2$. Then

$$\frac{\partial f}{\partial x} = -3\,x^2 + 2\,x\lambda\,z + 2\,zx - \lambda\,z^2,$$
$$\frac{\partial f}{\partial y} = 2yz,$$
$$\frac{\partial f}{\partial z} = y^2 + x^2 - 2\,x\lambda\,z + x^2\lambda.$$

When $\lambda = 0$, there is an ordinary double singularity at $[0:0:1]$. When $\lambda = 1$, there is an ordinary double singularity at $[1:0:1]$. The curve is smooth when $\lambda \neq 0, 1$.

(ii) This curve has no singularities.

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Q-6) For each $\lambda \in \mathbb{C}$, find the singular points and the multiplicities and the nature (i.e. ordinary or not) of the singular points of the following projective curve.

$$x^3 + y^3 + z^3 + 3\lambda xyz = 0.$$

Answer:

Let

$$f(x, y, z) = x^3 + y^3 + z^3 + 3\lambda xyz = 0.$$

To find the singular points, we must solve the system of equations

$$\begin{aligned} f(x, y, z) &= x^3 + y^3 + z^3 + 3\lambda xyz = 0, \\ f_x(x, y, z) &= 3x^2 + 3\lambda yz = 0, \\ f_y(x, y, z) &= 3y^2 + 3\lambda xz = 0, \\ f_z(x, y, z) &= 3z^2 + 3\lambda xy = 0. \end{aligned}$$

If $\lambda = 0$, there is no solution in $\mathbb{P}^2$. We observe that

$$xf_x = yf_y = zf_z, \text{ or equivalently } x^3 = y^3 = z^3.$$

Since $[x : y : z :]$ is a point in $\mathbb{P}^2$, we may assume that $x = 1$. Then $y, z \in \{1, w, w^2\}$, where w is a primitive cube root of unity. Solving for λ from $f_x = 0$, we find

$$\lambda = -\frac{x^2}{yz}, \text{ and } \lambda^3 = -1.$$

So we have that $\lambda \in \{-1, -w, -w^2\}$. Only for these values of λ, the curve will be singular. For all other values of λ, the curve is smooth.

To find the nature of singularities, observe that

$$\begin{aligned} (x + y + z)(x + wy + w^2z)(x + w^2y + wz) &= x^3 + y^3 + z^3 + 3(-1)xyz, \\ (x + wy + z)(x + w^2y + w^2z)(x + y + wz) &= x^3 + y^3 + z^3 + 3(-w)xyz, \\ (x + w^2y + z)(x + wy + wz)(x + y + w^2z) &= x^3 + y^3 + z^3 + 3(-w^2)xyz. \end{aligned}$$

Thus the curve becomes a union of three distinct lines. The singular points are the intersections of these lines, which are respectively given by the list

$$\begin{array}{ccc} (1, 1, 1), & (1, w, w^2), & (1, w^2, w), \\ (w, 1, 1), & (1, w, 1), & (1, 1, w), \\ (w^2, 1, 1), & (1, w^2, 1), & (1, 1, w^2). \end{array}$$