Date: 24 April 2012, Tuesday
Time: 13:40-15:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 431 Algebraic Geometry – Midterm Exam II – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.

We work over the complex numbers.

This is an open book, open notebook exam.

NAME: STUDENT NO:

Q-1) Let C be a compact Riemann surface of genus 2. Assume that the Riemann-Roch theorem holds for C.

1. Show that there exists a non-constant holomorphic function $f : C \to \mathbb{P}^1$ such that $f^{-1}(\infty) = p + q$ for some $p, q \in C$ with $p \neq q$.
2. Why did we not take C as a non-singular curve in $\mathbb{P}^2$ where we already know that the Riemann-Roch theorem holds?

Solution:

1. Since $\ell(K) = g = 2$, there is a nonconstant meromorphic function f such that $D = (f) + K \geq 0$. Clearly $D \sim K$ and $\deg D = \deg K = 2g - 2 = 2$, so we might as well take $K = p + q$. Let g be a nonconstant meromorphic function in $\mathcal{L}(K)$. If g has a pole only at p or only at q, then g gives an isomorphism between our Riemann surface and $\mathbb{P}^1$ and the genus becomes zero. Hence g must have a pole at p and q, and is the required degree two map onto $\mathbb{P}^1$.
2. From the degree-genus formula for projective plane curves, we know that there is no genus 2 curve in the plane.

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Q-2)

(i) Construct a canonical divisor on $\mathbb{P}^1$. Is it holomorphic or meromorphic?

(ii) Show that there are no holomorphic differentials on $\mathbb{P}^1$.

(iii) Show that a divisor of the form $p - q$ cannot be a principal divisor on any non-singular algebraic curve with positive genus, where $p \neq q$.

Solution:

(i) Let homogeneous coordinates on $\mathbb{P}^1$ be $[x : y]$ and let U be the open set with $x \neq 0$ and V with $y \neq 0$. On U we have the local coordinates $X = y/x$ and on V we have $Y = x/y$, with transition function given by $X = 1/Y$. Let

$$\omega = dX = -\frac{1}{Y^2}dY.$$

Then a canonical divisor can be constructed as

$$K = -2[0 : 1].$$

(ii) Let α be a holomorphic differential on $\mathbb{P}^1$. Then (α) is a canonical divisor and $\deg(\alpha) \geq 0$ since α has no poles. But the degree of a canonical divisor must be $2g - 2 = -2$ on $\mathbb{P}^1$. This contradiction shows that a holomorphic divisor cannot exist on $\mathbb{P}^1$.

(iii) If $(f) = p - q$, then f defines an isomorphism onto $\mathbb{P}^1$ forcing the genus to be zero.

NAME: STUDENT NO:

Q-3) Give a proof the Riemann-Roch theorem on $\mathbb{P}^1$ using elementary complex calculus.

Solution:

Let D be a divisor on $\mathbb{P}^1$. If $\deg D < 0$, then $\ell(D) = 0$ from the definition of $\mathcal{L}(D)$. So assume $\deg D \geq 0$.

Let K be a canonical divisor on $\mathbb{P}^1$. We found in the previous question that $\deg K = -2$. Then $\deg(K - D) < 0$ and $\ell(K - D) = 0$.

First assume that $\deg D = 0$.

If $D = 0$, then $\mathcal{L}(D) \simeq \mathbb{C}$ and $\ell(D) = 1$.

If $D \neq 0$, then we can easily construct a rational function having the prescribed poles and zeros. Any two such functions are constant multiples of each other, so again $\ell(D) = 1$.

Next assume that $\deg D > 0$.

Let $D = D_n = D_0 + p_1 + \cdots + p_n$ where $\deg D_0 = 0$. We claim that $\ell(D_n) = n + 1$. We prove this by induction. For $n = 0$, we just proved it above. Assume that the statement is true for $n - 1$. We have two cases to consider for the nature of p_n. If p_n occurs already with negative coefficient in D_0, then $D_{n-1} + p_n$ can again be written as $D_0 + p_1 + \cdots + p_{n-1} + q_n$ with a different divisor D_0 of degree 0 and with some point $q_n \in \mathbb{P}^1$ which we can again denote by p_n. This bring the problem to the content of the next case.

Now assume that p_n does not appear with negative coefficient in D_{n-1}. Let its coefficient in D_{n-1} be m where $m \geq 0$. Let $f \in \mathcal{L}(D_n)$ but not in $\mathcal{L}(D_{n-1})$. Then f has a pole of order $m + 1$ at p_n. Let g be another such function in $\mathcal{L}(D_n)$ but not in $\mathcal{L}(D_{n-1})$.

Let z be a local coordinate centered at p_n. Then the Laurent expansion of both of these functions are of the form

$$\frac{b_m}{z^{m+1}} + \frac{b_{m-1}}{z^m} + \cdots ,$$

where $b_m \neq 0$. Then there is a $\lambda \in \mathbb{C}$ such that the order of pole of $g - \lambda f$ at p_n is strictly less than $m + 1$, hence $g - \lambda f$ lies in $\mathcal{L}(D_{n-1})$. This shows that $\ell(D_n) = \ell(D_{n-1}) + 1$, which completes the induction.

Thus we proved that $\ell(D) = \deg(D) + 1$ on $\mathbb{P}^1$ with the understanding that $\ell(D) = 0$ if $\deg D < 0$.

NAME: STUDENT NO:

Q-4) Let C be a non-singular projective plane curve. Let D be a divisor on C. Choose any $p \in C$.

(i) Show that $\mathcal{L}(D) \subset \mathcal{L}(D+p)$.

(ii) Assuming that $\ell(D)$ is finite, show that $\ell(D+p) \leq \ell(D)+1$.

(iii) Show that $\ell(D)$ is finite.

Solution:

(i) Let $f \in \mathcal{L}(D)$. Then $(f)+D \geq 0$ so it trivially follows that $(f)+D+p \geq 0$ and hence $f \in \mathcal{L}(D+p)$.

(ii) If $\mathcal{L}(D) = \mathcal{L}(D+p)$, then there is nothing to prove. Assume that there exists a function $f \in \mathcal{L}(D+p)$ but not in $\mathcal{L}(D)$. Let m be the coefficient of p in D. Then the order of vanishing of f at p is $m+1$. As in the previous question, any two such functions are constant multiples of each other. Hence $\ell(D+p) \leq \ell(D)+1$.

(iii) The proof of this basically follows the idea of the previous part. Examining different cases in detail we show that $\ell(D+p) \leq \ell(D)+1$, and hence $\ell(D) \leq \deg D+1$ when $\deg D \geq 0$. There is clearly nothing to prove when $\deg D < 0$.

NAME: STUDENT NO:

Q-5) Let $C \subset \mathbb{P}^2$ be a non-singular algebraic curve. Let a line L intersect C at the distinct points p_1, p_2, p_3, p_4, p_5.
Choose a point $a \in C - \{p_1, p_2, p_3, p_4, p_5\}$.
Let L_i be the line joining a to p_i, and let D_i be the divisor consisting of the points $C \cap L_i$, counting multiplicities, $i = 1, \ldots, 5$.
Let $D = D_1 + \cdots + D_5 - 5a - p_1 - \cdots - p_5$.
Calculate $\ell(D)$.

Solution:

From Bezout's theorem we conclude that $\deg C = 5$, and from degree-genus formula we find that the genus g of C is 6. If K is a canonical divisor, it follows from $\deg K = 2g - 2$ that $\deg K = 10$.

We find that $\deg D = 15$ so that $\deg(K - D) < 0$ and hence $\ell(K - D) = 0$.

Now from Riemann-Roch theorem we find that $\ell(D) = \deg D + 1 - g = 15 + 1 - 6 = 10$.