Due Date: February 24, 2014 Monday NAME:..

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Math 431 Algebraic Geometry – Homework 1 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) (General Mathematics)
What are Fields, Abel and Gauss prizes? Who were the most recent recipients? What are IMU, MSRI and IHES?

Answer:

The official web page for Fields Medals is Fields Medals

The official web page for Abel prize is Abel Prize

A link for the Gauss Prize is Gauss Prize

The web page for IMU, International Mathematical Union, is IMU

The web page for MSRI, Mathematical Sciences Research Institute, is MSRI

The web page for IHES, Institut des Hautes Études Scientifiques, is IHES

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Q-2) (Topology)

Show that in a Noetherian topological space, every non-empty closed set can be expressed as a finite union of irreducible closed sets, unique up to permutation and up to redundancy.

Solution:

Let X be a non-empty closed subset of a Noetherian topological space. If X is irreducible, then we are done. If not, then we can write X as a union of irreducible closed subsets. Suppose that this union is infinite; $X = X_1 \cup X_2 \cup \cdots$. By defining $Y_t = X_t \cup X_{t+1} \cup \cdots$, $t = 1, 2, \ldots$, we obtain a non-terminating descending chin of closed subsets, violating the Noetherian property of the space. Thus we can write $X = X_1 \cup \cdots \cup X_n$, where $X_i \not\subset X_j$ if $i \neq j$. Suppose we can also write $X = Y_1 \cup \cdots \cup Y_m$, where Y_i are irreducible closed subsets and $Y_i \not\subset Y_j$ when $i \neq j$. We have $X_1 = X_1 \cap X = X_1 \cap (Y_1 \cup \cdots \cup Y_m) = X_1 \cap Y_i$ for some i since X_1 is irreducible. Without loss of generality assume that $X_1 = X_1 \cap Y_1$. So $X_1 \subset Y_1$. Similarly $Y_1 \subset X_i$ for some i, but this gives $X_1 \subset X_i$, forcing $i = 1$ and hence $X_1 = Y_1$. By induction, and by re-indexing if necessary, we see that $n = m$ and $X_i = Y_i$ for $i = 1, \ldots n$. This completes the proof.

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Q-3) (Commutative Algebra)

Show, using only a sketch of ideas, that there exists a Noetherian ring with infinite (Krull) dimension. You can find such an example on page 203 of Nagata's book *Local Rings* (1962). For understanding this example you will need to learn what it means to localize a ring at a multiplicatively closed set.

Solution:

The basic idea is simple: put together noetherian rings of finite dimension n for each positive integer n. The resulting ring is clearly noetherian but is of infinite dimensional. The challenge however is in implementing this idea. I refer to the above entry in Nagata's book for the beautiful construction and for the proof that it works.

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Q-4) (Algebraic Geometry)

Let k be an algebraically closed field of characteristic $p \geq 0$ but $p \neq 2$. Let $f \in k[x, y]$ be an irreducible quadratic polynomial. How many different (i.e. non-isomorphic) $Z(f) \subset \mathbb{A}^2$ does there exist? What about $p = 2$ case?

Solution:

We first assume that the underlying field k is algebraically closed and that $\operatorname{char} k \neq 2$. Let $f(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + Fz$ be the given irreducible quadratic polynomial and let $g(x, y, z) = z^2 f(x/z, y/z)$ be its homogenization with respect to z. First we find criteria for the irreducibility of f. Clearly f is irreducible if and only if g is. To g corresponds a matrix

$$M_f = \begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix}$$

which can be diagonalized after a base change. Let $g(u, v, w) = c_1 u^2 + c_2 v^2 + c_3 w^2$ with respect to this new base. It can be shown easily that $g(u, v, w)$ is a product of linear forms if and only if at least one of the c_i's is zero or equivalently if $\det M_f = 0$. Hence $f(x, y)$ is irreducible if and only if $\det M_f \neq 0$.

Next we consider the ring $A(W) = k[x, y]/(f)$. For this consider the change of bases given by

$$\begin{aligned} x &= \alpha u - \beta v \\ y &= \beta u + \alpha v \end{aligned}$$

where α and β are in k with

$$\alpha^2 + \beta^2 = 1,$$

which describes a 'rotation'. This transfers $f(x, y)$ into $f(u, v) = A'u^2 + B'uv + C'v^2 + D'u + E'v + F'$, where $B' = B(\alpha^2 - \beta^2) + 2(C - A)(\alpha\beta)$. Since k is algebraically closed we can choose α and β to make $B' = 0$. Then $A(W)$, which is now isomorphic to $k[u, v]/(f(u, v))$, is isomorphic to $k[x]$ if either $A' = 0$ or $C' = 0$, and is isomorphic to $k[x, \frac{1}{x}]$ otherwise. Noting that $B^2 - 4AC = (B')^2 - 4A'C'$ we can reformulate this as

$$A(W) \cong \begin{cases} k[x] & \text{if } B^2 - 4AC = 0; \\ k[x, \frac{1}{x}] & \text{if } B^2 - 4AC \neq 0. \end{cases}$$

When $\operatorname{char} k = 2$, the story is different and involves Arf invariant.