Due Date: April 21, 2014 Monday

NAME:....................................................

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STUDENT NO:....................................................

## Math 431 Algebraic Geometry – Homework 2 – Solutions

| 1 | 2 | 3 | 4 | TOTAL |
|---|---|---|---|---|
| | | | | |
| 50 | 50 | - | - | 100 |

*Please do not write anything inside the above boxes!*

Check that there are **2** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

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NAME: STUDENT NO:

**Q-1)** Let $G$ be an Arf semigroup and $a < b < c$ be three consecutive elements in $G$, i.e. the only element of $G$ in the open real interval $(a, c)$ is $b$. Show that $c - b \leq b - a$, i.e. the elements of $G$ get closer. Show that this is not necessarily the case for every semigroup.

**Answer:**

Let the integers

$$i_0 = 0, i_1, i_2, \dots, i_h, \dots$$

form an Arf semigroup, where $i_n < i_{n+1}$ for every $n \geq 0$. For any $h \geq 0$, since the integers

$$i_h - i_h = 0, i_{h+1} - i_h, i_{h+2} - i_h, \dots,$$

form a semigroup, they are closed under addition. In particular we must have

$$2(i_{h+1} - i_h) \geq i_{h+2} - i_h.$$

It follows that

$$i_{h+1} - i_h \geq i_{h+2} - i_{h+1},$$

which is what we want to show.

To show that this is not always the case if $G$ is not an Arf ring, let $G$ be the semigroup generated by 3 and 5;

$$G = \{0, 3, 5, 6, 8, 9, 10 \dots\}.$$

Let $a = 5$, $b = 6$, $c = 8$. Then $c - b = 2 \not\leq b - a = 1$.

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**Q-2)** Let $G = \{5m + 7n \mid m, n \in \mathbb{N}\}$. Show that the complement of $G$ in $\mathbb{N}$ is finite. Find the Frobenius number of $G$, i.e. the largest integer not in $G$. Construct ${}^*G$, the Arf closure of $G$. Find the generators of ${}^*G$.
Find the multiplicity sequence of the plane cusp $y^5 = x^7$. How does this sequence relate to the elements of ${}^*G$?

**Solution:**

We have

$$G = \{0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, \overline{24}\},$$

where $\overline{24}$ means $24+n$ for every nonnegative integer $n$. Then $\mathbb{N}\backslash G = \{1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23\}$. Thus the Frobenius number is 23. Let $G_1$ be the semigroup generated by the differences $7-5, 10-5, 12-5, \ldots$. We see that

$$G_1 = \{0, 2, \overline{4}\}.$$

We know that ${}^*G = \{0, 5 + {}^*G_1\}$. But $G_1$ is clearly an Arf ring, so

$${}^*G = \{0, 5 + G_1\} = \{0, 5, 7, \overline{9}\}.$$

Therefore the generators of ${}^*G$ are 5, 7, 9, 11 and 13.

On the other hand if we blow up the singularity $y^5 = x^7$, we get in turn the singularities $y^5 = x^2$, $y^3 = x^2$ and finally the singularity is resolved and we get $y = x^2$. This gives the multiplicity sequence 5,2,2,1,1,.... The associated ring is

$$0,\ 5,\ 5+2,\ 5+2+2,\ 5+2+2+1,\ 5+2+2+1+1, \ldots$$

which is precisely ${}^*G$.