Final Exam for Math 431 **Due Date: 20 May 2022**
Solution Key

1) In an appendix to *Introduction to Algebraic Curves*, Griffiths sketches a proof of the following statement.

> If C is a compact complex Riemann surface, then there exists an immersion of C into $\mathbb{P}^2$ such that $f(C)$ is an algebraic curve with at most double points as singularities.

Since he uses this result for his development of the theory, he has to prove this using elementary methods.

Prove this result now using whatever we learned so far about Riemann surfaces and algebraic geometry.

Sketch of Solution: We showed that if D is a divisor of C with $\deg D \geq 2g+1$, where g is the genus of C, then a basis of $L(D)$ gives an immersion of C into some $\mathbb{P}^n$.

Let P be a point in $\mathbb{P}^n$ not lying on any tangent or secan line of C. Then a projection from P to a linear subspace $\mathbb{P}^{n-1}$ will embed C into $\mathbb{P}^{n-1}$. Repeating this we end up in $\mathbb{P}^3$. Here the set of all secants of C may fill all of $\mathbb{P}^3$. However we can still choose a 'nice' point Q in $\mathbb{P}^3$ such that projection from Q onto $\mathbb{P}^2$ will send C to a plane curve with only double points. Details of how to choose this point Q are given in Hartshorne's Algebraic Geometry on pages 309-314. Any reasonable summary is accepted.

2) Let C be a smooth cubic curve in $\mathbb{P}^2$, the ground field being $\mathbb{C}$. For any $p, q \in C$, let L be the line through p and q when $p \neq q$, and be the tangent line to C at p when $p = q$. By Bezout's theorem we have $L \cdot C = p + q + r$ for some $r \in C$. This defines a map $\phi \colon C \times C \to C$ as $\phi(p, q) = r$, where r is defined as above. Fix a point $p_0 \in C$. Define $p \oplus q$ for any $p, q \in C$ as

$$p \oplus q = \phi(p_0, \phi(p, q)).$$

Show that:

(i) $p \oplus q = q \oplus p$ for any $p, q \in C$

(ii) $p_0 \oplus p = p$ for any $p \in C$.

(iii) For every $p \in C$ there exists a $q \in C$ such that $p \oplus q = p_0$.

(iv) $p \oplus (q \oplus r) = (p \oplus q) \oplus r$ for any $p, q, r \in C$.

Thus C is an abelian group under this operation.

Solution: First observe that by definition $\phi(p, q) = \phi(q, r)$. Hence

$$p \oplus q = \phi(p_0, \phi(p, q)) = \phi(p_0, \phi(q, p)) = q \oplus p.$$

For the second part observe again by definition that p_0, p and $\phi(p_0, p)$ are on the same line. Thus

$$p_0 \oplus p = \phi(p_0, \phi(p_0, p)) = p.$$

For the third part let L_0 be the tangent line to C at p_0, and let q_0 be the third point where L_0 intersects C. For any $p \in C$ let $\phi(p, q_0) = q$. This means that p, q_0, q are on the same line, so $\phi(p, q) = q_0$. Now we have

$$p \oplus q = \phi(p_0, \phi(p, q)) = \phi(p_0, q_0) = p_0.$$

The associativity property takes a bit of calculation to show. Here we go!

Given $p, q, r \in C$.

Let L_1 be the line through p and q.
$L_1 \cdot C = p + q + s'$, where $s' = \phi(p, q)$.

Let M_1 be the line through p_0 and s'.
$M_1 \cdot C = p_0 + s' + s$, where $s = \phi(p_0, s') = \phi(p_0, \phi(p, q)) = p \oplus q$.

Let L_2 be the line through s and r.
$L_2 \cdot C = s + r + t'$, where $t' = \phi(s, r)$.
Then by definition $(p \oplus q) \oplus r = \phi(p_0, t')$. ♠

Now let M_2 be the line through q and r.
$M_2 \cdot C = q + r + u2$, where $u' = \phi(q, r)$.
Let L_3 be the line through p_0 and u'.
$L_3 \cdot C = p_0 + u' + u$, where $u = \phi(p_0, u')$.

Let M_3 be the line through p and u.
$M_3 \cdot C = p + u + t''$, where $t'' = \phi(p, u)$.
Then by definition $p \oplus (q \oplus r) = \phi(p_0, t'')$. ♠

So now it suffices to show that $t' = t''$.

Let $C' = L_1 L_2 L_3$ and $C'' = M_1 M_2 M_3$

$C \cdot C' = C \cdot L_1 + C \cdot L_2 + C \cdot L_3 = p + q + s' + s + r + t' + p_0 + u' + u$,
$C \cdot C'' = C \cdot M_1 + C \cdot M_2 + C \cdot M_3 = p_0 + s' + s + q + r + u' + p + u + t''$.

At this point we need magic!

Cayley-Bacharach Theorem: Let C and D be two curves in $\mathbb{P}^2$ of degrees m and n respectively, meeting at mn points. Let E be another curve of degree $m+n-3$ passing through all but one point of $C \cap D$. Then E passes through that remaining point also.

Taking $m = n = 3$, we see that if two cubic curves C and C' intersect at 9 points and C'' passes through 8 of these, then it passes from the remaining point too.

In our notation above this says that $t = t'$, completing the proof of associativity.