Midterm-1 for Math 431 **Due Date: 1 April 2022**

Your Take-Home Exam assignment consists of three exercises from Hartshorne's book: Exercises I.5.1, I.5.2 and I.5.7.

These are reproduced below for your convenience. You may find several solutions on the Internet but first attack these problems on your own. In particular for I.5.7 first try $f(x, y, z) = x^2 + y^2 + z^2$ and draw the real picture. This will tell you what is happening in general. Check what happens when $\deg f = 1$. After playing with equations on your own you can then search the Internet and talk with your friends.

Write your solutions on A4 papers and staple them together. Do not use other fancy covers!

Thank you.

I.5.1. **Locate the singular points and sketch the following curves in $\mathbb{A}^3$ (assume char $k \neq 2$). Which is which in Figure 4?**
(a) $x^2 = x^4 + y^4$;
(b) $xy = x^6 + y^6$;
(c) $x^3 = y^2 + x^4 + y^4$;
(d) $x^2y + xy^2 = x^4 + y^4$.

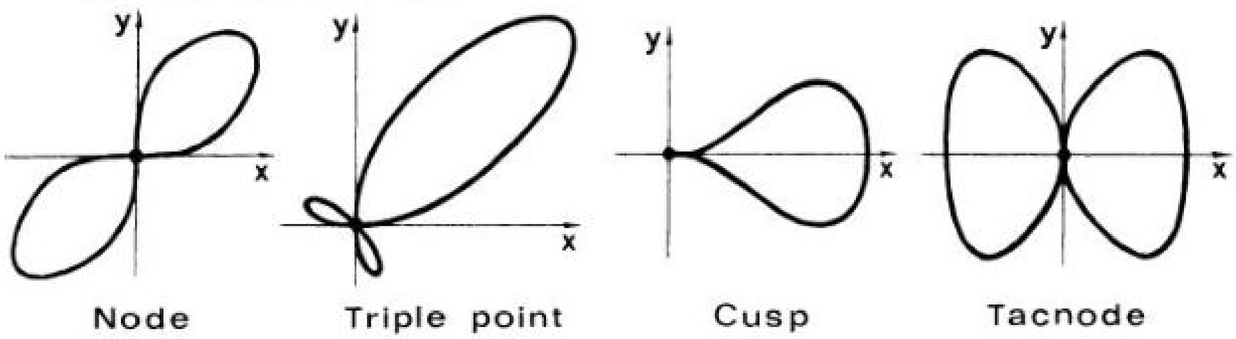

Figure 4. Singularities of plane curves.

To distinguish among these curves we examine the so called tangent cones of the given equations. The tangent cone is the zero set of the smallest degree homogeneous part of the given polynomial and this gives the different tangential directions at the singularity which is the origin in this case.

(a)
$x^2 = 0$ gives a double tangent along the y-axis. This is then a tacnode.

(b)
$xy = 0$ gives the x and y-axes as tangents. This is then a node.

(c)
$y^2 = 0$ gives x-axis as the double tangent. This is then cusp.

(d)
$x^2y + xy^2 = xy(x + y) = 0$ gives x and y-axes as tangents in addition to the line $x + y = 0$. This is then a triple point.

I.5.2. Locate the singular points and describe the singularities of the following surfaces in $\mathbb{A}^3$ (assume char $k \neq 0$). Which is which in Figure 5?
(a) $xy^2 = z^2$;
(b) $x^2 + y^2 = z^2$;
(c) $xy + x^3 + y^3 = 0$.

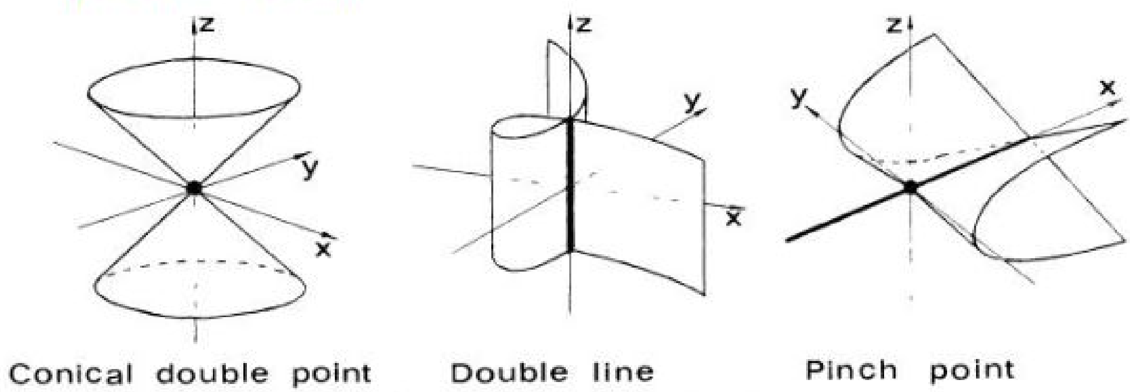

Figure 5. Surface singularities.

Here given $f(x, y, z) = 0$ we find the points on the surface where the Jacobian $J = (f_x, f_y, f_z)$ vanishes.

(a)
Here $f(x, y, z) = z^2 - xy^2$. Then $J = (-y^2, -2xy, 2z) = (0, 0, 0)$ gives the

x-axis as the only solution which also lies on the surface. This is given as a pinch point.

(b)
Here $f(x, y, z) = x^2 + y^2 - z^2$. Then $J = (2x, 2y, -2z) = (0, 0, 0)$ gives the origin as the only singularity. This is given as a conical double point.

(c)
Here $f(x, y, z) = xy + x^3 + y^3$. Then $J = (y + 3x^2, x + 3y^2, 0) = (0, 0, 0)$ together with $f = 0$ gives the z-axis as the only singularity. This is then a double line.

I.5.7. Let $Y \subseteq \mathbb{P}^n$ be a nonsingular plane curve of degree > 1, defined by the equation $f(x, y, z) = 0$. Let $X \subseteq \mathbb{A}^{n+1}$ be the affine variety defined by f (this is the cone over Y). Let p be the point $(0, 0, 0)$, which is the vertex of the cone. Let $\phi \colon \tilde{X} \to X$ be the blow-up of X at p.
(a) Show that X has just one singular point, namely p.
(b) Show that $\tilde{X}$ is nonsingular.
(c) Show that $\phi^{-1}(p)$ is isomorphic to Y.

(a)
Since Y is smooth, for any point $q = [q_0 : q_1 : q_2] \in Y$, we have

$$J(q) = \left(\frac{\partial f}{\partial x}(q_0, q_1, q_2), \frac{\partial f}{\partial y}(q_0, q_1, q_2), \frac{\partial f}{\partial z}(q_0, q_1, q_2) \right) \neq (0, 0, 0).$$

But each of these partial derivatives is a homogeneous polynomial of degree $= d - 1 > 0$, so $p = (0, 0, 0)$ gives

$$J(p) = (0, 0, 0),$$

and hence p is the only singular point of X.

(b)
We can write

$$\tilde{X} = \{(x, y, z) \times [r : s : t] \in \mathbb{A}^3 \times \mathbb{P}^2 \mid xs = yr,\ xt = zr,\ yt = zs,\ f(x, y, z) = 0\}.$$

On the coordinate chart $r \neq 0$ we have new coordinates

$$U = x,\ V = \frac{s}{r},\ W = \frac{t}{r},$$

and hence the change of coordinates between (x, y, z) and (U, V, W) are given as

$$x = U,\ y = UV,\ z = UW.$$

Then we have

$$f(x, y, z) = f(U, UV, UW) = U^d f(1, V, W) = 0,$$

where $U = 0$ corresponds to the exceptional divisor and $f(1, V, W) = 0$ corresponds to the equation of Y in the open coordinate chart $x \neq 0$ in $\mathbb{P}^2$. Hence $\tilde{X}$ is nonsingular in the chart $r \neq 0$. By symmetry $\tilde{X}$ is nonsingular in the other charts too.

(c)
As shown above $\phi^{-1}(p)$ corresponds to $x = 0$ and $[r : s : t] \in \mathbb{P}^2$ free, and $r \neq 0$ in our chart. Then

$$0 = f(1, V, W) = f(1, \frac{s}{r}, \frac{t}{r}) = \frac{f(r, s, t)}{r^d},$$

which forces

$$f(r, s, t) = 0,$$

which is the equation of Y in the chart $r \neq 0$. Hence $\phi^{-1}(p) \cong Y$.