Homework-1 for Math 431 **Due Date: 1 March 2022**

Your are expected to solve Exercise I.2.6 on page 11 of Hartshorne, *Algebraic Geometry*. The exercise, quoted below, comes with its own hints. Moreover I include below a detailed version of that hint, which is almost a solution.

Your mission is to write in detail a solution, explaining all the steps you use in such a way that when you read your solution at the end of the semester you will understand and appreciate your solution. This means you do not leave out crucial arguments and do not include unnecessary details.

I.2.6 **If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$. [*Hint*: Let $\phi_i : U_i \to \mathbb{A}^n$ be the homeomorphism of (2.2), let Y_i be the affine variety $\phi_i(Y \cap U_i)$, and let $A(Y_i)$ be its affine coordinate ring. Show that $A(Y_i)$ be identified with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Now use (1.7), (1.8A), and (Ex 1.10), and look at transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is nonempty.]**

Idea of the solution:
Let $\tilde{Y}_i = Y \cap U_i$, and $Y_i = \phi_i(\tilde{Y}_i)$, $i = 0, \ldots, n$. Since Y is the union of the open subsets $\tilde{Y}_i$, by Exercise I.1.10.b, $\dim Y = \sup \dim \tilde{Y}_i$. Since there are only finitely many $\tilde{Y}_i$, we must have $\dim Y = \max \dim \tilde{Y}_i$. By notational convenience set $\dim Y = \dim \tilde{Y}_0$.

Since ϕ_0 is a homeomorphism, $\dim \tilde{Y}_0 = \dim Y_0$. By Proposition I.1.7, the dimension of Y_0 is equal to the Krull dimension of its affine coordinate ring $A(Y_0)$. Moreover by Theorem I.1.8A.a, the dimension of $A(Y_0)$ is equal to the transcendence degree of the field of quotients $K(A(Y_0))$ of the ring $A(Y_0)$.

It remains to determine the transcendence degree of $K(A(Y_0))$. For this we set the following notation. $S(Y)$ is the quotient ring of the graded ring $k[x_0, \ldots, x_n]$ by the homogeneous ideal of Y. $A(Y_0)$ is the quotient ring of the polynomial ring $k[y_1, \ldots, y_n]$ by the affine ideal of Y_0.

Following the hint of the problem we prove that the local ring $S(Y)_{x_0}$ is isomorphic to the Laurent ring $A(Y_0)[x_0, x_0^{-1}]$. A typical element of $S(Y)_{x_0}$

can be represented by the rational function

$$\frac{F(x_0, \ldots, x_n)}{x_0^d},$$

where $F(x_0, \ldots, x_n)$ is a homogeneous polynomial in $k[x_0, \ldots, x_n]$ of degree e, and d is a non-negative integer. We define a map

$$\lambda : S(Y)_{x_0} \longrightarrow A(Y_0)[x_0, x_0^{-1}]$$

by the formula

$$\lambda\left(\frac{F(x_0, \ldots, x_n)}{x_0^d}\right) = F(1, y_1, \ldots, y_n)x_0^{e-d}.$$

It is straightforward to show that

$$F(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})x_0^e = F(x_0, \ldots, x_n).$$

In fact, let $f(y_1, \ldots, y_n) \in k[y_1, \ldots, y_n]$ be a polynomial of degree e, and let ℓ be any integer. Then $f(y_1, \ldots, y_n)x_0^\ell$ represents a typical element of $A(Y_0)[x_0, x_0^{-1}]$. Next define a homogeneous polynomial F as

$$F(x_0, \ldots, x_n) = f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})x_0^e.$$

It then follows that the association

$$f(y_1, \ldots, y_n)x_0^\ell \mapsto F(x_0, \ldots, x_n)x_0^{\ell-e}$$

is the inverse of the map λ.

We can clearly force λ to obey ring operations and it then becomes a ring isomorphism. This establishes the fact that the rings $S(Y)_{x_0}$ and $A(Y_0)[x_0, x_0^{-1}]$ are isomorphic. This also shows that $A(Y_0)$ can be identified with the elements of degree zero of $S(Y)_{x_0}$, which corresponds to the case when $e = d$. But this observation is not used in the solution of this exercise.

It is easy to see that the field of quotients $K(S(Y))$ is the same as the field of quotients $K(S(Y)_{x_0})$. If we now let tr. deg denote transcendence degree over k, we have

$$\begin{aligned} \operatorname{tr.deg} K(S(Y)) &= \operatorname{tr.deg} K(S(Y)_{x_0}) \\ &= \operatorname{tr.deg} K(A(Y_0)[x_0, x_0^{-1}]) \\ &= \operatorname{tr.deg} K(A(Y_0)) + 1. \end{aligned}$$

This gives in turn, using Theorem I.1.8A.a, that

$$\dim S(Y) = \dim A(Y_0) + 1.$$

Finally, recalling that $\dim A(Y_0) = \dim Y_0 = \dim \tilde{Y_0} = \dim Y$, we get

$$\dim S(Y) = \dim Y + 1,$$

as claimed.

In the above proof we actually proved that if Y_i is not empty, then $\dim Y_i = \dim A(Y_i) = \dim S(Y) - 1$. Since this last expression does not depend on the index i, it follows that $\dim Y = \dim Y_i$ whenever Y_i is not empty.

Important: A homework is part of learning, not a chore to be submitted and got over with. You can learn only if you take the homework problems seriously.

This is what you should do:

Try to solve the problem on your own to see how far you go. Then talk with your friends, or with me, and get all the help you need. Then start writing your solution *alone*. If you get stuck again go out and get help. When you resolve the point where you got stuck, go back and continue to write your solution again *alone*. Repeat as necessary.

Never, ever, lend your written work, and ask to borrow someone else's written work until after all the homeworks are submitted. After that you can compare your written works with each other to complete the learning benefits of the homework.