Homework-3 for Math 431 **Due Date: 15 April 2022 Friday**

Let $f, g \in k[x_0, \ldots, x_n]$ be irreducible homogeneous polynomials. Write the Hilbert polynomial of $Z(f, g) \subset \mathbb{P}^n$.

Write your solutions on A4 papers and staple them together. Do not use other fancy covers!

Thank you.

Solution:
Let Y_a and Y_b are hypersurfaces in $\mathbb{P}^3$ of degrees a and b respectively with $Y_a = Z(f)$ and $Y_b = Z(g)$ where f and g are homogeneous polynomials of degrees a and b. Assume that $Y = Y_a \cap Y_b$ is a complete intersection, i.e. the two hypersurfaces Y_a and Y_b have no common components.

From the proof of Proposition 7.6, on page 52, we know the Hilbert polynomials of Y_a and Y_b; for example

$$P_{Y_a}(z) = \binom{z+n}{n} - \binom{z-a+n}{n}.$$

We now consider the short exact sequence of graded S-modules

$$0 \longrightarrow (S/f))\,(-b) \xrightarrow{g} S/f \longrightarrow S/(f,g) \longrightarrow 0,$$

where S is the graded polynomial ring $k[x_0, \ldots, x_n]$, the first map is multiplication by g, and the second map is the natural quotient map. Noting that S/f and $S/(f,g)$ are the projective coordinate rings of Y_a and Y respectively, we have from the additivity of the Hilbert function

$$\begin{aligned} P_Y(z) &= P_{Y_a}(z) - P_{Y_a}(z-b) \\ &= \binom{z+n}{n} - \binom{z-a+n}{n} - \left[\binom{z-b+n}{n} - \binom{z-b-a+n}{n}\right]. \end{aligned}$$