Due Date: 18 October 2012, Thursday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 503 Complex Analysis – Exam 01

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Describe the Riemann surface of the image of $f(z) = \cos z$.

Solution:

$\cos(x + iy) = u(x, y) + iv(x, y)$ where $u(x, y) = \cos x \cosh y$ and $v(x, y) = \sin x \sinh y$. Consider the w-plane as $w = u + iv$.

Any strip $S_k = \{x + iy \in \mathbb{C} \mid k\pi < x < (k + 1)\pi,\ y \in \mathbb{R}\}$ is mapped injectively onto the w plane except the cuts along the u-axis described by $u \leq -1$ and $u \geq 1$.

The line segment $(k\pi, (k+1)\pi)$ of the z-plane is mapped onto the line segment $(-1, 1)$ of the w-plane. Its direction is from $+1$ to -1 if k is even, and reversed if k is odd.

Observe that for any $y_0 \neq 0$, we have

$$\frac{u^2(x, y_0)}{\cosh^2 y_0} + \frac{v^2(x, y_0)}{\sinh^2 y_0} = \cos^2 x + \sin^2 x = 1.$$

This describes an ellipse. Depending on the sign of y_0 and the parity of k, the line segment $-k\pi < x < (k + 1)\pi$ and $y = y_0$ is mapped to the lower or upper part of this ellipse in the w-plane.

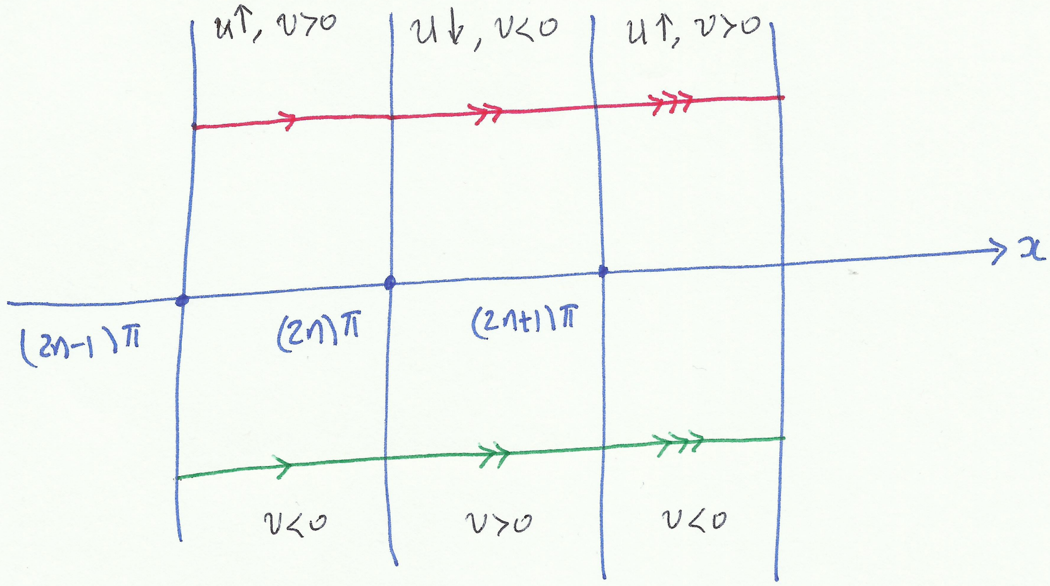
To make this precise, choose $y_0 > 0$ and let $L_k(y_0) = \{x + iy_0 \mid k\pi < x < (k + 1)\pi\}$. For $x + iy_0 \in L_k(y_0)$, we have $u(x, y_0) = \cos x \cosh y_0$. When k is even it goes from $+1$ down to -1, end points excluded, and reverses direction when k is odd. On the other hand $v(x, y_0) = -\sin x \sinh y_0$. This is negative when k is even, and is positive when k is odd.

We follow the closures of the lines $L_k(y_0)$ consecutively as k increases and keep in mind that the image must also be a connected curve. This dictates how the sheets should be glued together.

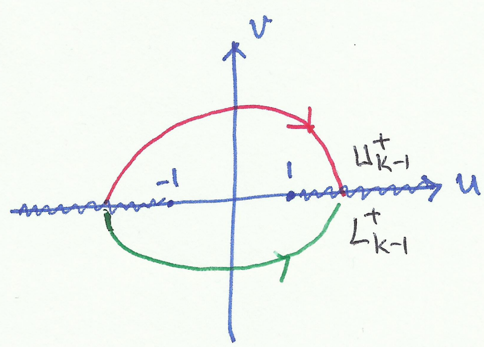
Let us denote by W_k a copy of the w-plane with cuts along the u-axis where $u \leq -1$ and $u \geq 1$. After we cut the plane, let U_k^+ denote the line along the cut for $u \geq 1$ belonging to the upper part of the cut, and L_k^+ be the lower part. Similarly define U_k^- and L_k^- for the cut along $u \leq -1$. Assume that k is even. Then the gluing of the sheets is performed as follows.

L_k^- is glued to U_{k+1}^-, and U_k^- is glued to L_{k+1}^-.

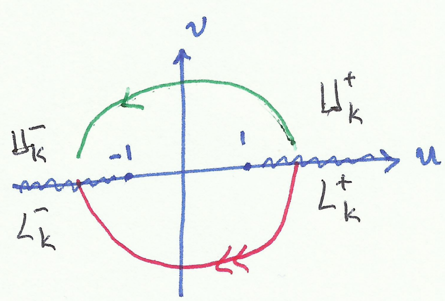
L_k^+ is glued to U_{k-1}^+, and U_k^+ is glued to L_{k-1}^+.



u↑, v>0
u↓, v<0
u↑, v>0
x
(2n-1)π
(2n)π
(2n+1)π
v<0
v>0
v<0

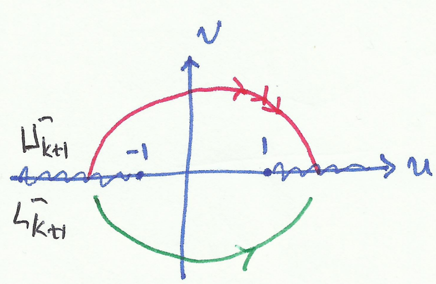


v
u
-1
1
U^+_{k-1}
L^+_{k-1}



v
u
-1
1
U^-_k
L^-_k
U^+_k
L^+_k

$k = 2n$



v
u
-1
1
U^-_{k+1}
L^-_{k+1}

$w = \cos z$

NAME: STUDENT NO:

Q-2) Describe the Riemann surface of the image of $f(z) = \tan z$.

Solution:

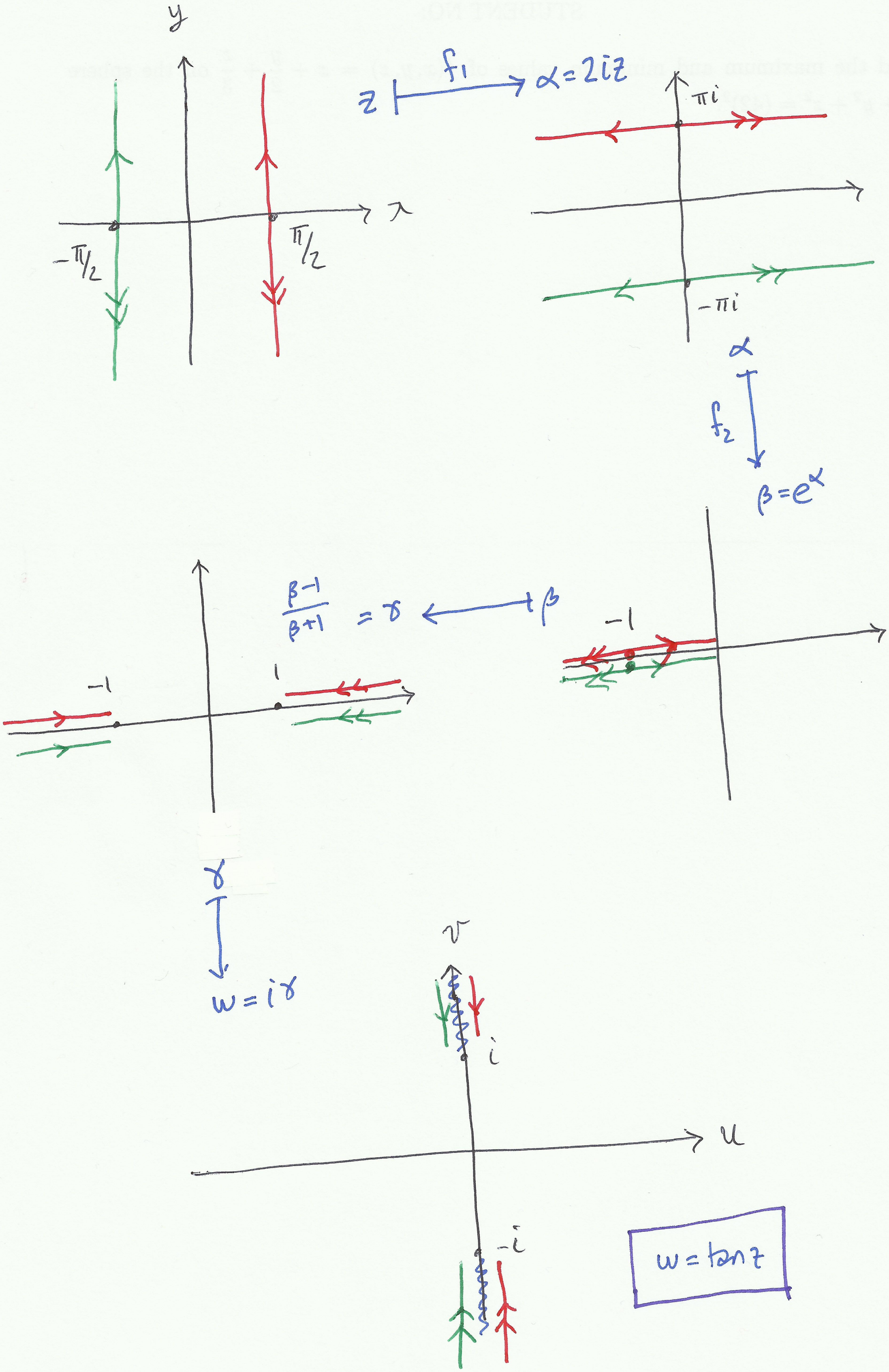
Recall that $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, and $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$. This gives $\tan z = -i\dfrac{e^{2iz} - 1}{e^{2iz} + 1}$.

We can therefore consider $z \longrightarrow w = \tan z$ as the composition of the following maps.

$$z \xrightarrow{f_1} \alpha = 2iz, \ \alpha \xrightarrow{f_2} \beta = e^{\alpha}, \ \beta \xrightarrow{f_3} \gamma = \frac{\beta - 1}{\beta + 1}, \ \gamma \xrightarrow{f_4} w = -i\gamma.$$

We start with the fundamental domain $-\dfrac{\pi}{2} \le \operatorname{Re} z \le \dfrac{\pi}{2}$, where we exclude $z = \pm\dfrac{\pi}{2}$.

As a result, the Riemann surface of $w = \tan z$, using the above fundamental region, can be described as infinitely many copies of the w-plane. Letting $w = u + iv$, each sheet is cut along $u = 0$ and $|v| \ge 1$. If the sheets are numbered by integers, then the right hand side of the cuts of the n-th sheet are glued to the left hand side cuts of the $n + 1$-st sheet, for $n \in \mathbb{Z}$.


y
x
$-\pi/2$
$\pi/2$
$z \overset{f_1}{\longmapsto} \alpha = 2iz$
πi
$-\pi i$
$\alpha \overset{f_2}{\longmapsto} \beta = e^{\alpha}$
-1
$\frac{\beta-1}{\beta+1} = \gamma \longleftarrow \beta$
-1
1
$\gamma \longmapsto w = i\gamma$
v
u
i
$-i$
$w = \tan z$

NAME: STUDENT NO:

Q-3) Describe the Riemann surface of the image of $f(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$. The superfluous looking constant $1/2$ is attached to normalize calculations.

Solution:

Taking $z = e^{i\theta}$, we see that if $w = u + iv$, yhen

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta,\ v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta.$$

Since z and $1/z$ are mapped to the same point, both the inside of the unit disc and its outside are mapped onto the complex plane minus a cut along $[-1, 1]$. The upper piece of the unit circle and the lower piece each are mapped onto this interval in a one to one manner.

So take two copies of the w-plane, each with a cut along the interval $[-1, 1]$. Glue them along this cut so that as you move down towards zero along the v-axis of one of the sheets, you pass to the negative v axis of the other sheet as you pass the origin. Call one of the sheets the lower sheet and the other the upper sheet for convenience.

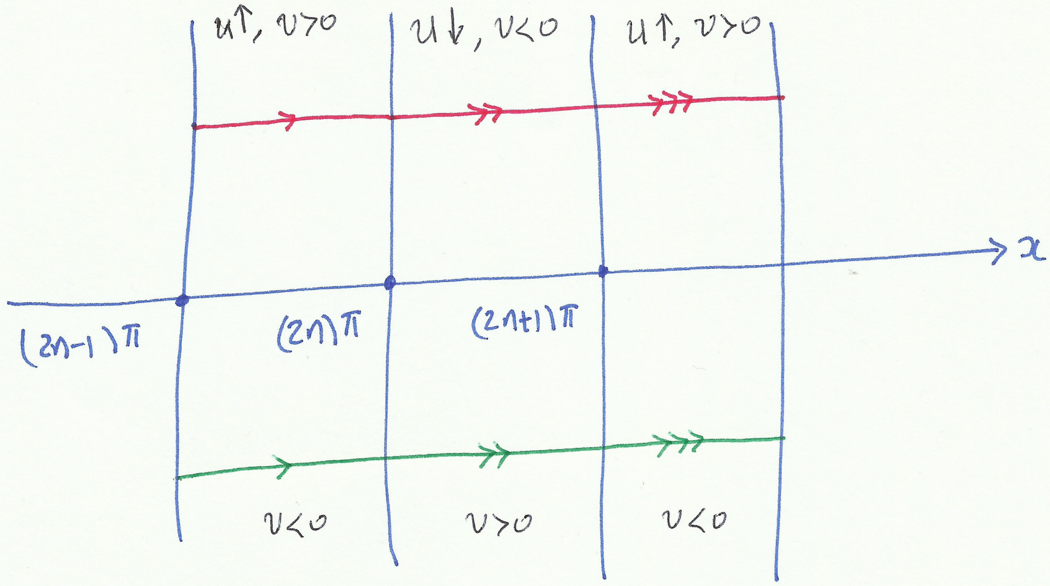
The inside of the unit disc is mapped onto one of these sheets and the outside is mapped onto the other sheet. For convenience let us agree that the inside of the unit disc is mapped onto the lower sheet and the outside is mapped onto the upper sheet.

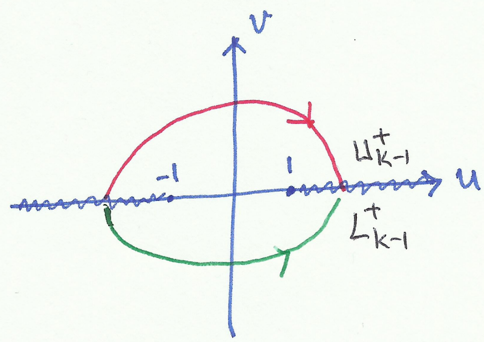
If L_θ is a ray from the origin, $r > 0$, making an angle θ with the x-axis, then its image satisfies the equation

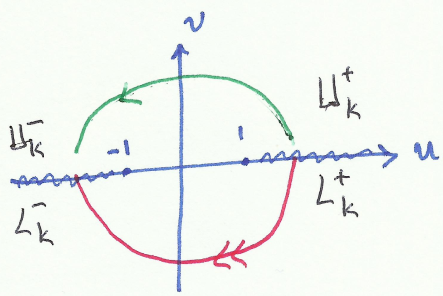
$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 1,$$

which is a hyperbola which intersects the u-axis at a point where $|u| < 1$. This u-intercept corresponds to the glued cut, so the hyperbola passes from one sheet to the other sheet along the cut.

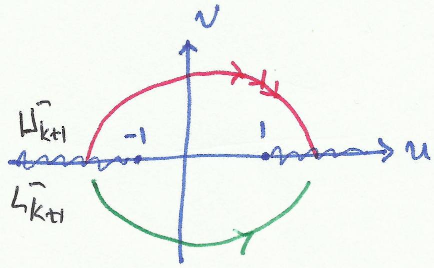
For each ray, its piece outside the unit disc maps into the upper sheet as an arm of a hyperbola. Its position is determined by the signs of $\cos\theta$ and $\sin\theta$, as in the above formulation for u and v. Similarly all ray pieces living inside the unit disc are mapped into the lower sheet as an arm of a hyperbola.


u↑, v>0
u↓, v<0
u↑, v>0
x
(2n-1)π
(2n)π
(2n+1)π
v<0
v>0
v<0


v
u
-1
1
U^+_{k-1}
L^+_{k-1}


v
u
-1
1
U^-_k
L^-_k
U^+_k
L^+_k

$k = 2n$


v
u
-1
1
U^-_{k+1}
L^-_{k+1}

$w = w_n z$

NAME: STUDENT NO:

Q-4) Show that for any polynomial $P(z)$, there exists $z \in \mathbb{C}$ with $|z| = 1$ such that

$$\left| P(z) - \frac{1}{z} \right| \geq 1.$$

Solution:

Assume to the contrary that there exists a polynomial $P(z)$ such that for every z on the unit circle we have

$$\left| P(z) - \frac{1}{z} \right| < 1.$$

Multiplying both sides of the inequality by $|z|$ where z is on the unit circle, we get

$$|zP(z) - 1| < 1,$$

for all z on the unit circle. From the maximum modulus theorem, we then have that

$$|zP(z) - 1| < 1,$$

for all z with $|z| \leq 1$. But putting $z = 0$ we get a contradiction. Hence our assumption is wrong, and the claim of the theorem is correct.

NAME: STUDENT NO:

Q-5) Let $D = \{z \in \mathbb{C} \mid |z| \leq 2\}$. Prove or disprove that for every $\epsilon > 0$ there exists a polynomial $P(z)$ such that for every non-zero $z \in D$

$$\left| P(z) - \frac{1}{z} \right| < \epsilon.$$

Solution:

The claim is wrong. Take $\epsilon = 1$. Using the previous question, we know that for every polynomial $P(z)$, there exists a point z on the unit disc, inside D, such that

$$\left| P(z) - \frac{1}{z} \right| \geq 1.$$