Due Date: 29 November 2012, Thursday NAME:..

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Math 503 Complex Analysis – Exam 06

1	2	3	4	5	TOTAL
45	45	10	0	0	100

Please do not write anything inside the above boxes!

Check that there are **3** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

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Q-1) Show that a meromorphic function cannot have three *independent* periods, in the following sense.

Let f be a meromorphic function and Z the integral module of its periods. Then only one of the following cases holds.

1. $Z = \{0\}$, i.e. f is not periodic.
2. There exists a non-zero $\omega_1 \in \mathbb{C}$ such that $Z = \{n\omega_1 \mid n \in \mathbb{Z}\}$. In this case f is called a periodic function.
3. There exist non-zero $\omega_1, \omega_2 \in \mathbb{C}$ such that $\omega_1/\omega_2 \notin \mathbb{R}$ and $Z = \{n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z}\}$. In this case f can also be called periodic, because it is, but traditionally it is called an elliptic function.

Solution: Since we have examples of each of the above cases what remains to be shown is that there cannot exist a periodic meromorphic function whose period module cannot be generated by two or less elements.

Assume that $Z \subset \mathbb{C}$ is an integral module which is the period module of some meromorphic function. If $Z = \{0\}$, we are done. Otherwise let $\omega_1 \in Z$ be a non-zero element such that $|\omega_1| \leq |\omega|$ for all $\omega \in Z - \{0\}$. If $Z' = Z - \{n\omega_1 \mid n \in \mathbb{Z}\}$ is empty, we are done. Otherwise let $\omega_2 \in Z'$ be such that $|\omega_2| \leq |\omega|$ for all $\omega \in Z'$.

Note that ω_1/ω_2 is not a real number. To show this assume that $\omega_1/\omega_2 = \lambda$ is real. Replacing ω_2 by $-\omega_2$ if necessary, we may assume that $\lambda > 0$. Then $\omega_1 = \lambda\omega_2$. Since $|\omega_1| \leq |\omega_2|$, we must have $\lambda \leq 1$, but since $\omega_2 \in Z'$, we cannot have $\lambda = 1$. So $0 < \lambda < 1$. Here $\lambda \neq 1/2$, since then we would have $\omega_2 = 2\omega_1 \notin Z'$ contradicting the choice of ω_2. Hence there exists an integer $n \geq 1$ such that $0 < 1 - n\lambda < \lambda$. Then $(1 - n\lambda)\omega_2 \in Z$ and $|(1 - n)\lambda\omega_2| < |\lambda\omega_2| = |\omega_1|$ contradicting the choice of ω_1. Therefore ω_1/ω_2 is not real.

Now if $Z'' = Z - \{n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z}\}$ is empty, we are done.

Assume Z'' is not empty. Let $\omega_3 \in Z''$ be such that $|\omega_3| \leq |\omega|$ for all $\omega \in Z''$. By the way we chose them, notice that $0 < |\omega_1| \leq |\omega_2| \leq |\omega_3|$. In particular Z'' contains no ω with $|\omega| < |\omega_2|$.

Since ω_1 and ω_2 are linearly independent over the reals, i.e. ω_1/ω_2 is not real, there exist real numbers λ_1 and λ_2 such that $\omega_3 = \lambda_1\omega_1 + \lambda_2\omega_2$. We can now choose integers n_1 and n_2 such that $|n_j - \lambda_j| \leq 1/2$, $j = 1, 2$. Define $\omega = (n_1\omega_1 + n_2\omega_2) - \omega_3 = (n_1 - \lambda_1)\omega_1 + (n_2 - \lambda_2)\omega_2 \in Z$. Clearly $\omega \in Z''$. But now we have $|\omega| = |(n_1 - \lambda_1)\omega_1 + (n_2 - \lambda_2)\omega_2| < |n_1 - \lambda_1||\omega_1| + |n_2 - \lambda_2||\omega_2| \leq |\omega_2|$, where the first inequality is strict since ω_1 and ω_2 are $\mathbb{R}$-linearly independent. But this contradicts the choice of ω_3, which forces Z'' to be empty.

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Q-2) Let f be an elliptic function with a fundamental domain $P = \{x\omega_1 + y\omega_2 \mid 0 \leq x, y \leq 1\}$. Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be the zeros and poles of f inside P, respectively, repeated according to multiplicity. We assume that none of the zeros and poles are on the boundary of P.

Show that there exists integers m and n such that

$$\sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i = n\omega_1 + m\omega_2.$$

Solution:

Ahlfors page 263:

We will calculate the integral

$$I = \frac{1}{2\pi i} \int_{\partial P} \frac{zf'(z)}{f(z)}\, dz$$

in two different ways.

By the generalized Cauchy Integral Formula, we immediately have

$$I = (a_1 + \cdots + a_k) - (b_1 + \cdots + b_k).$$

Next we notice that because of periodicity that integrals along opposite edges of P can be related to each other and can be easily calculated. For example

$$\frac{1}{2\pi i} \int_0^{\omega_1} \frac{zf'(z)}{f(z)}\, dz - \frac{1}{2\pi i} \int_{\omega_2}^{\omega_1+\omega_2} \frac{zf'(z)}{f(z)}\, dz = -\frac{\omega_2}{2\pi i} \int_0^{\omega_1} \frac{f'(z)}{f(z)}\, dz.$$

Note however that $\frac{i}{2\pi i} \int_0^{\omega_1} \frac{f'(z)}{f(z)}\, dz$ is the winding number of the curve $w = f(t\omega_1)$, $t \in [0, 1]$, around the origin in the w-plane. Hence this integral is an integer, say m.

Similarly we see that

$$\frac{1}{2\pi i} \int_0^{\omega_2} \frac{zf'(z)}{f(z)}\, dz - \frac{1}{2\pi i} \int_{\omega_1}^{\omega_1+\omega_2} \frac{zf'(z)}{f(z)}\, dz = -\frac{\omega_1}{2\pi i} \int_0^{\omega_2} \frac{f'(z)}{f(z)}\, dz = n\omega_1,$$

for some integer n. This completes the proof.

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Q-3) State, but do not prove, how all elliptic functions are related to Weierstrass $\mathfrak{p}$-functions.

Solution:

Let f be an elliptic function with fundamental periods ω_1 and ω_2. Then there exists a rational function $g(x, y)$ such that $f(z) = g(\mathfrak{p}(z), \mathfrak{p}'(z))$, where $\mathfrak{p}$ is the Weierstrass $\mathfrak{p}$ function with the fundamental periods ω_1 and ω_2.