Due Date: 28 December 2012, Friday NAME:..

Please leave your homework in my mailbox until 17:30.

Ali Sinan Sertöz STUDENT NO:..

Math 503 Complex Analysis – Exam 10

1	2	3	4	5	TOTAL
50	50	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** question on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

The following two problems are taken from Conway's *Functions of One Complex Variable*, page 185, Exercises 3 and 5, respectively.

Q-1) Show that $\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$. Using this find an expression for $\Gamma(n+\frac{1}{2})$ for all integers $n \geq 0$.

Solution: Define a new function

$$F(z,N) = \frac{N!\,N^z}{z(z+1)(z+2)\cdots(z+N)},$$

where z is a complex variable with $\operatorname{Re} z > 0$ and N is a positive integer. Gauss' formula says that

$$\Gamma(z) = \lim_{N\to\infty} F(z,N).$$

In particular the limit on the right exists, so we can take the limit along any subsequence of the integers going to infinity and obtain the same limit. We will use the following limit.

$$\frac{\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)} = \lim_{m\to\infty} \frac{F(z,2m)F(z+\frac{1}{2},2m)}{F(2z,4m+1)}.$$

At this point we employ Stirling's formula in the following form.

$$N! = \lim_{N\to\infty} \sqrt{2\pi N}\left(\frac{N}{e}\right)^N E_N, \quad \text{where} \quad \lim_{N\to\infty} E_N = 1.$$

Substituting every factorial with its form from the Stirling's formula and simplifying, we find

$$\frac{F(z,2m)F(z+\frac{1}{2},2m)}{F(2z,4m+1)} = \frac{2\sqrt{\pi}}{\left(2+\frac{1}{2m}\right)^{2z}}\tilde{E}_m,$$

where $\lim_{m\to\infty} \tilde{E}_m = 1$. (*see appendix for details.*) Now taking limits of both sides as $m \to \infty$, we find

$$\frac{\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)} = \frac{\sqrt{\pi}}{2^{2z-1}},$$

which is precisely what we wanted to show.

Finally, putting $z = n$, we get $\Gamma(n+\dfrac{1}{2}) = \dfrac{\sqrt{\pi}\,(2n-1)!}{2^{2n-1}(n-1)!} = \dfrac{\sqrt{\pi}\,(2n)!}{4^n n!}$.

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Q-2) Let f be analytic on the right half plane $\operatorname{Re} z > 0$ and satisfy: $f(1) = 1$, $f(z+1) = zf(z)$, and $\lim_{n\to\infty} \frac{f(z+n)}{n^z f(n)} = 1$ for all z. Show that $f = \Gamma$.

Solution: The condition $f(z+1) = zf(z)$ implies that $f(z+n) = z(z+1)\cdots(z+n-1)f(z)$, where n is a positive integer..

The condition $f(1) = 1$, together with the above expression implies that $f(n) = (n-1)!$, where again n is a positive integer.

Now we take the last condition.

$$\begin{aligned} \frac{f(z+n)}{n^z f(n)} &= \frac{z(z+1)\cdots(z+n-1)f(z)}{n^z (n-1)!} \\ &= \frac{z(z+1)\cdots(z+n-1)(z+n)}{n^z n!} \frac{n}{z+n} f(z). \end{aligned}$$

Taking limits of both sides as $n \to \infty$, and using Gauss' formula together with the third condition of the problem, we find

$$1 = \frac{f(z)}{\Gamma(z)},$$

which is precisely what we wanted to show.

$$\frac{F(z,2m)\,F(z+\frac{1}{2},2m)}{F(2z,4m+1)} = \frac{(2m)!\,(2m)^{z}\;(2m)!\,(2m)^{z+\frac{1}{2}}\cdot 2z(2z+1)\cdots(2z+4m+1)}{z(z+1)\cdots(z+2m)\cdot(z+\frac{1}{2})(z+\frac{1}{2}+1)\cdots(z+\frac{1}{2}+2m)\cdot(4m+1)!\,(4m+1)^{2z}}$$

$$= \underbrace{\frac{(2m)!\,(2m)!}{(4m+1)!}}_{\gamma_m}\;\underbrace{\frac{(2m)^{z}(2m)^{z+\frac{1}{2}}}{(4m+1)^{2z}}}_{\beta_m}\cdot\underbrace{\frac{2\cancel{z}(2z+1)\cdots(2z+4m+2)}{\cancel{z}\underbrace{(z+1)\cdots(z+2m)}_{2m\text{ terms}}\underbrace{(z+\frac{1}{2})(z+\frac{1}{2}+1)\cdots(z+\frac{1}{2}+2m)}_{2m+1\text{ terms}}}}_{\alpha_m}$$

$$\alpha_m = \frac{2\cancel{z}(2z+1)(2z+2)\cdots(2z+4m+1)}{\cancel{z}\underbrace{(z+1)(z+2)\cdots(z+2m)}_{2m\text{ terms}}\cdot\underbrace{(z+\frac{1}{2})(z+\frac{1}{2}+1)(z+\frac{1}{2}+2)\cdots(z+\frac{1}{2}+2m)}_{2m+1\text{ terms}}}\cdot\frac{2^{4m+1}}{2^{4m+1}}$$

$$= 2^{4m+2}\,\frac{(2z+1)(2z+2)\cdots(2z+4m+1)}{(2z+2)(2z+4)\cdots(2z+4m)\cdot(2z+1)(2z+3)\cdots(2z+4m+1)} = 2^{4m+1}$$

$$\beta_m = \frac{(2m)^z \, (2m)^{z+\frac{1}{2}}}{(4m+1)^{2z}} = \frac{(2m)^{2z+\frac{1}{2}}}{(4m+1)^{2z}} = \frac{\sqrt{2}\,\sqrt{m}}{\left(2 + \frac{1}{2m}\right)^{2z}}$$

$$\gamma_m = \frac{(2m)!\,(2m)!}{(4m+1)!} = \lim_{m\to\infty} \frac{\sqrt{2\pi(2m)}\left(\frac{2m}{e}\right)^{2m} \cdot E_{2m} \cdot \sqrt{2\pi(2m)}\left(\frac{2m}{e}\right)^{2m} \cdot E_{2m}}{(4m+1)\sqrt{2\pi(4m)}\left(\frac{4m}{e}\right)^{4m} E_{4m}}$$

$$= \lim_{m\to\infty} \frac{2\pi \cdot 2m \cdot (2m)^{4m} \cdot \boxed{E_{2m} \cdot E_{2m}}}{(4m+1)\sqrt{2\pi}(2\sqrt{m}) \; (4m)^{4m} \boxed{E_{4m}}} \to \text{call this } \tilde{E}_m$$

$$= \lim_{m\to\infty} \frac{2m}{4m+1} \cdot \sqrt{2\pi}\,\frac{1}{\sqrt{m}} \cdot \frac{1}{2^{4m}}, \qquad \tilde{\gamma}_m = \frac{2m}{4m+1} \cdot \sqrt{2\pi}\,\frac{1}{m^{1/2}} \cdot \frac{1}{2^{4m}}$$

$$\tilde{\gamma}_m \beta_m \alpha_m = \left(\frac{2m}{4m+1} \cdot \sqrt{2\pi} \; \frac{1}{\cancel{m^{1/2}}} \cdot \frac{1}{\cancel{2^{4m}}} \right) \left(\frac{\sqrt{2} \cdot \cancel{m^{1/2}}}{\left(2+\frac{1}{2m}\right)^{2z}} \right) \left(\cancel{2^{4m+1}} \; 2 \right)$$

$$\lim_{m\to\infty} \tilde{\gamma}_m \beta_m \alpha_m = \frac{1}{2} \sqrt{2\pi} \; \sqrt{2} \cdot \frac{1}{2^{2z}} \cdot 2 = \frac{\sqrt{\pi}}{2^{2z-1}} .$$

Also $\lim\limits_{m\to\infty} \dfrac{F(z,2m)\, F(z+\frac{1}{2},2m)}{F(2z,4m+1)} = \dfrac{\Gamma(z)\,\Gamma(z+\frac{1}{2})}{\Gamma(2z)}$.

qed :)