Due Date: 16 November 2017, Thursday
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:.......................................

Math 503 Complex Analysis - Homework 3 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

General Rules for Take-Home Assignments

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you **write your answers alone**.

(2) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(3) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you **exhibit your total understanding of the ideas involved**.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

Please sign here:

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Q-1) Show that $e^{\frac{1}{z}}$ has an essential singularity at $z = 0$.

Solution:

The easiest way is to check that the Laurent series of $e^{\frac{1}{z}}$ has infinitely many negative powers of z.

However we can show that $z = 0$ is an essential singularity bu using the definition directly.

Let $z_n = \dfrac{1}{2n\pi i}$. Then $\lim_{n\to\infty} e^{1/z_n} = 1$.

Now let $z_n = \dfrac{1}{(2n+1)\pi i}$. Then $\lim_{n\to\infty} e^{1/z_n} = -1$.

This shows that $\lim_{z\to 0} e^{1/z}$ does not exist and hence $z = 0$ is an essential singularity by definition.

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Q-2) Some details of the proof of Lemma 5.1 on page 83 is left to the reader. Fill out these details. (This lemma is crucially used in developing the theory of Laurent series on page 107.)

Solution:

The only detail left to the reader is to supply the second line below and to observe that the third line below is as given in the book.

$$\begin{aligned}
\frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} &= \left[\frac{1}{w-z} - \frac{1}{w-a}\right] \sum_{k=1}^{m} \frac{1}{(w-z)^{m-k}} \frac{1}{(w-a)^{k-1}} \\
&= \left[\frac{z-a}{(w-z)(w-a)}\right] \sum_{k=1}^{m} \frac{1}{(w-z)^{m-k}} \frac{1}{(w-a)^{k-1}} \\
&= (z-a) \sum_{k=1}^{m} \frac{1}{(w-z)^{m-k+1}} \frac{1}{(w-a)^{k}}
\end{aligned}$$

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Q-3 Solve Exercise 10 on page 111, using what we developed so far. Note that the result trivially follows from Great Picard Theorem on page 300.

Solution:

Here is the complete statement of the exercise:

> Suppose that f has an essential singularity at $z = a$. Prove the following strengthened version of the Casorati-Weierstrass Theorem. If $c \in \mathbb{C}$ and $\epsilon > 0$ are given then for each $\delta > 0$ there is a number α, $|c - \alpha| < \epsilon$, such that $f(z) = \alpha$ has infinitely many solutions in $B(a; \delta)$.

Given $\epsilon, \delta > 0$ and $c \in \mathbb{C}$ define

$$B_n := \{z \in \mathbb{C} \mid 0 < |z - a| < \delta/n\},\ n = 1, 2, \ldots,$$

and let

$$U_n := f(B_n),\ n = 1, 2, \ldots.$$

Each U_n is open by the open mapping property of analytic functions, and is dense by the Casorati-Weierstrass Theorem. Since $\mathbb{C}$ with the usual Euclidean metric is complete, the Baire category property holds: every union of a countable family of open dense sets is dense in $\mathbb{C}$. Let

$$U := \bigcap_{n=1}^{\infty} U_n.$$

Then U is dense in $\mathbb{C}$ and let $\alpha \in U$ be such that $|c - \alpha| < \epsilon$.

Since $\alpha \in U_1$, there exists a number $z_1 \in B_1$ such that $f(z_1) = \alpha$. Clearly $z_1 \in B(a; \delta)$.

Having chosen $z_1, \ldots, z_k \in B(a; \delta)$, choose $z_{k+1} \in B_m$ such that $f(z_{k+1}) = \alpha$ and

$$\delta/m < \min\{|z_1|, \ldots, |z_k|\}.$$

Clearly z_k is different than each of $z_1, \ldots, z_k$, and $z_k \in B(a; \delta)$.

Thus $z_1, z_2, \ldots$ constitute an infinite collection of solutions to $f(z) = \alpha$.

Q-4) Find the value of $\displaystyle\int_0^\infty \frac{(\log x)^4}{1+x^2}\,dx$.

Solution:

We first note that $\displaystyle\int_0^1 \frac{(\log x)^{2n+1}}{1+x^2}\,dx$ becomes $\displaystyle -\int_1^\infty \frac{(\log t)^{2n+1}}{1+t^2}\,dt$ after change of variables given by $x = 1/t$. Hence

$$\int_0^\infty \frac{(\log x)^{2n+1}}{1+x^2}\,dx = 0, \text{ for every integer } n \geq 0.$$

The $n = 0$ case is explicitly done using residues, on page 117 as Example 2.10 of Conway.

Now define

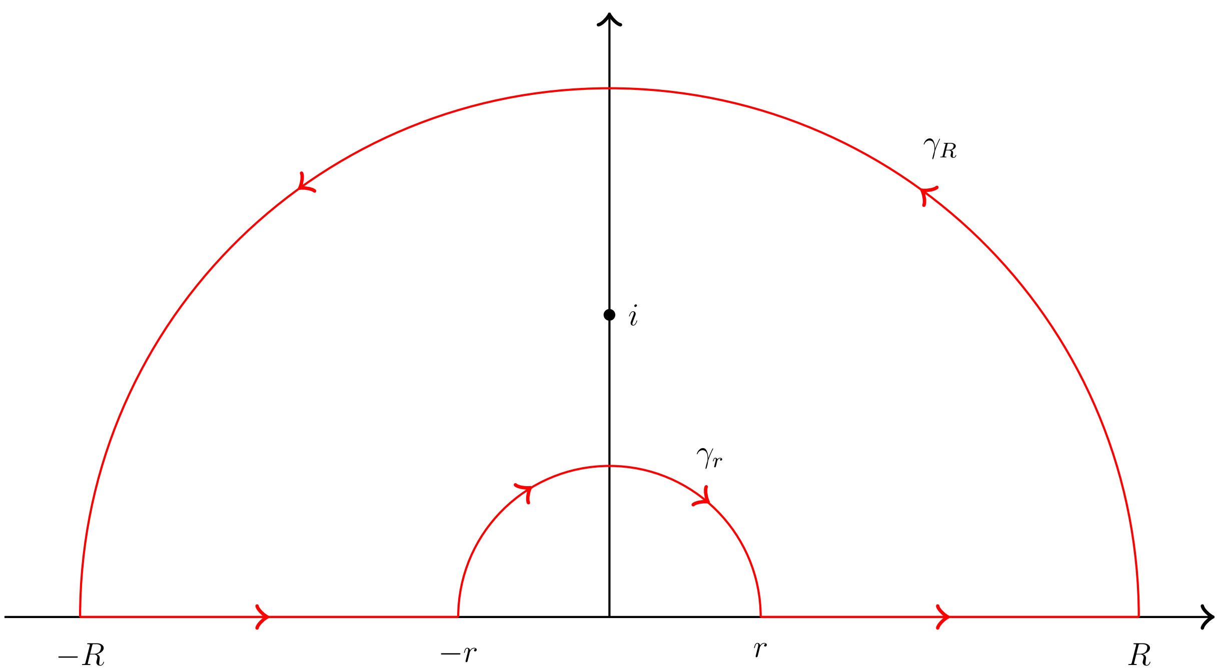
$$I_n := \int_0^\infty \frac{(\log x)^n}{1+x^2}\,dx, \text{ for every integer } n \geq 0.$$

We showed above that $I_n = 0$ for odd n. We also know that

$$I_0 = \frac{\pi}{2}, \quad I_2 = \frac{\pi^3}{8}.$$

The first follows from calculus and the second was done in class. Now we want to calculate I_4.

For this we will use the path $C_{r,R} = \gamma_R + [-R, -r] + \gamma_r + [r, R]$, as shown below, where $0 < r < 1 < R$.



Let $f(z) = \dfrac{(\log z)^4}{1+z^2}$. There is only one singularity of f inside this path at $z = i$. This is a simple pole so we can easily calculate the residue.

$$\text{Res}(f, i) = \frac{(\log i)^4}{2i} = \frac{(\frac{\pi}{2}\,i)^4}{2i} = \frac{\pi^4}{32i}.$$

Hence we have, for every r, R satisfying $0 < r < 1 < R$,

$$\int_{C_{r,R}} f(z)\,dz = 2\pi i\,\mathrm{Res}(f, i) = \frac{\pi^5}{16}.$$

Now we calculate the left hand side separately.

(1) On γ_ρ for some $\rho > 0$.

For $z \in \gamma_\rho$ we have $z = \rho e^{i\theta}$ with $0 \le \theta \le \pi$, and $dz = \rho i e^{i\theta}\,d\theta$. We thus have after parametrization

$$\begin{aligned}
\left| \int_{\gamma_\rho} \frac{(\log z)^4}{1+z^2}\,dz \right| &= \left| \int_0^\pi \frac{(\log(\rho e^{i\theta}))^4}{1+\rho^2 e^{i2\theta}}\,\rho i e^{i\theta}\,d\theta \right| \\
&\le \int_0^\pi \frac{\rho|\log\rho + i\theta|^4}{|1-\rho^2|}\,d\theta \\
&\le \frac{\rho(\log\rho)^4}{|1-\rho^2|}\int_0^\pi d\theta + \frac{4\rho(\log\rho)^3}{|1-\rho^2|}\int_0^\pi \theta\,d\theta + \frac{6\rho(\log\rho)^2}{|1-\rho^2|}\int_0^\pi \theta^2\,d\theta \\
&\quad + \frac{4\rho(\log\rho)}{|1-\rho^2|}\int_0^\pi \theta^3\,d\theta + \frac{4\rho}{|1-\rho^2|}\int_0^\pi \theta^4\,d\theta
\end{aligned}$$

We see that the right hand side goes to zero when ρ goes both to zero and to infinity. Thus we have

$$\lim_{r\to 0^+}\int_{\gamma_r} f(z)\,dz = 0, \text{ and } \lim_{R\to\infty}\int_{\gamma_r} f(z)\,dz = 0.$$

(2) On $[r, R]$ we have $z = x$, $dz = x$ and we have

$$\int_{[r,R]} f(z)\,dz = \int_r^R \frac{(\log x)^4}{1+x^2}\,dx.$$

(3) On $[-R, -r]$ we have $z = -x = xe^{i\pi}$ and $dz = -dx$ for $r \le x \le R$. We now have

$$\frac{(\log z)^4}{1+z^2} = \frac{(\log x + i\pi)^4}{1+x^2} = \frac{(\log x)^4}{1+x^2} + \frac{4\pi i(\log x)^3}{1+x^2} - \frac{6\pi^2(\log x)^2}{1+x^2} - \frac{4\pi i(\log x)}{1+x^2} + \frac{\pi^4}{1+x^2}.$$

Note that

$$\int_{[-R,-r]} f(z)\,dz = \int_r^R \frac{(\log x + i\pi)^4}{1+x^2}\,dx.$$

(4) Putting all these together, taking the limit as $r \to 0$ and $R \to \infty$, and recalling the known values of I_n, we get

$$\frac{\pi^5}{16} = \lim_{\substack{r\to 0\\ R\to\infty}} \int_{C_{r,R}} f(z)\,dz = 2I_4 - 6\pi^2 I_2 + \pi^4 I_0,$$

and finally

$$I_4 = \frac{5\pi^5}{32}.$$