Due Date: 19 October 2017, Thursday

NAME:..

STUDENT NO:..

Math 503 Complex Analysis - Midterm 1 – Solution Key

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

General Rules for Take-Home Assignments

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you **write your answers alone**.

(2) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(3) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you **exhibit your total understanding of the ideas involved**.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

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NAME: STUDENT NO: DEPARTMENT:

Q-1) Let G be simply connected open subset of $\mathbb{C}$ and $u(x, y)$ a harmonic function on G. Show that a harmonic conjugate for u exists on G.

Solution:

We need the Green's Theorem from Calculus: If $P(x, y)$ and $Q(x, y)$ are C^1 functions on a simply connected domain U and $\omega = Pdx + Qdy$ is a closed form, then the line integral of ω on paths totally lying in U are path independent.

Now let $u(x, y)$ be a harmonic function in the simply connected region U. Define the function $g(z) = u_x - iu_y$. Check that g is holomorphic in U.

Note that $g(z)dz = (u_x - iu_y)(dx + idy) = \omega_1 + i\omega_2$ where ω_1, ω_2 are real and closed. Hence, fixing a point $z_0 \in U$ we can define a function

$$F(z) = \int_{z_0}^{z} g(s)ds, \ z \in U.$$

Now fix any $z \in U$ and for a given $\epsilon > 0$ let $\delta > 0$ be such that whenever $|z - z'| < \delta$, we have $|g(z) - g(z')| < \epsilon$. Choose Δz such that $|\Delta z| < \delta$. Then we have

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - g(z)\right| = \left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z}(g(s) - g(z))ds\right| \leq \left|\frac{1}{\Delta z}\right| \int_{z}^{z+\Delta z} |g(s) - g(z)||ds| < \epsilon,$$

assuming we perform the last path independent integral along a line joining z to $z + \Delta z$; this line will stay in U for small $|\Delta z|$ since U is open. Now this last expression above shows that $F' = g$. Note

$$F(z)' = U_x + iV_x = V_y - iU_y = u_x - iu_y.$$

Then $U + c = u$ for some real constant c. Hence u is the real part of the holomorphic function $F + c$ and hence V is a harmonic conjugate for u.

NAME: STUDENT NO: DEPARTMENT:

Q-2) Derive, in your own words, the polar form of the Cauchy-Riemann equations and use that to show that the $\log$ function is holomorphic on $\mathbb{C} \setminus \{z \leq 0\}$.

Solution:

Let $f(z) = u(x, y) + iv(x, y)$ be a holomorphic function. We have the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Now keeping in mind that

$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta,$$

we take partial derivatives of u and v with respect to r and θ using the chain rule.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}(\cos\theta) + \frac{\partial u}{\partial y}(\sin\theta) = \frac{\partial v}{\partial y}(\cos\theta) - \frac{\partial v}{\partial x}(\sin\theta)$$

where we implemented the Cauchy-Riemann equations in the last line. Similarly we have

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial u}{\partial x}(r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) = -\frac{\partial v}{\partial y}(r\sin\theta) - \frac{\partial v}{\partial x}(r\cos\theta).$$

Comparing these with

$$\begin{aligned}
\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}(\cos\theta) + \frac{\partial v}{\partial y}(\sin\theta) \\
\frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial v}{\partial x}(r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)
\end{aligned}$$

we observe that we have the relations

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r},$$

which are the polar forms of the Cauchy-Riemann equations.

Finally we apply this to the $\log$ function. If $z = re^{\theta}$, then

$$\log z = \log r + i\theta, \quad \text{so} \quad u = \log r \quad \text{and} \quad v = \theta.$$

We immediately have

$$u_r = \frac{1}{r}, v_\theta = 1, u_\theta = 0, v_r = 0.$$

It then follows that the polar from of the Cauchy-Riemann equations hold for the $\log$ function.

NAME: STUDENT NO: DEPARTMENT:

Q-3 Classify all holomorphic functions on a fixed connected open subset of $\mathbb{C}$ which take only real values.

Solution:

Let $f(z) = u(x, y) + iv(x, y)$. Since f is taking only real numbers we must have $v \equiv 0$. From the Cauchy-Riemann equations we find that $u_x = 0$ and $u_y = 0$, so U is constant, since the domain is connected.

Thus the only holomorphic functions which take only real numbers on a connected domain are the constants.

NAME: STUDENT NO: DEPARTMENT:

Q-4) Calculate the following numbers, always using the fundamental branch of the logarithm whenever a logarithm is required.

(i) i^i

(ii) $(-2)^i$.

(iii) $\pi^{\frac{1+i}{1-i}}$

(iv) $\left(\sqrt{3}+i\right)^{1+i\sqrt{3}}$

(v) Let $a_0 = i$ and $a_n = (a_{n-1})^i$ for $n = 1, 2, \dots$. Find a_n.

Solution:

If $z = re^{i\theta}$, then the fundamental branch is chosen as $-\pi < \theta \leq \pi$.

(i) $i = e^{i\frac{\pi}{2}}$. $i^i = \exp(i \log i) = \exp(i\, i\frac{\pi}{2}) = e^{-\frac{\pi}{2}} \approx 0.20$.

(ii) First note that $2^i = \exp(i \ln 2) = \cos \ln 2 + i \sin \ln 2$.
Hence $(-2)^i = (2e^{i\pi})^i = 2^i e^{-\pi} = e^{-\pi}(\cos \ln 2 + i \sin \ln 2) \approx 0.03 + i0.02$.

(iii) First note that $\frac{1+i}{1-i} = i$. Hence $\pi^i = \exp(i \ln \pi) = \cos \ln \pi + i \sin \ln \pi \approx 0.41 + i0.91$.

(iv) $\left(\sqrt{3}+i\right)^{1+i\sqrt{3}} = (2e^{i\frac{\pi}{6}})^{1+i\sqrt{3}} = \exp((1+i\sqrt{3})(\ln 2 + i\dfrac{\pi}{6}) = \exp\left((\ln 2 - \dfrac{\sqrt{3}\pi}{6}) + i(\sqrt{3}\ln 2 + \dfrac{\pi}{6})\right)$.

This finally gives $\left(\sqrt{3}+i\right)^{1+i\sqrt{3}} = 2e^{-\frac{\sqrt{3}\pi}{6}}\left(\cos(\sqrt{3}\ln 2 + \dfrac{\pi}{6}) + i\sin(\sqrt{3}\ln 2 + \dfrac{\pi}{6})\right) \approx -0.12 + i0.79$.

(v) We check that the first few terms of the sequence are

$$a_0 = i, a_1 = e^{-\frac{\pi}{2}}, a_2 = -i, a_3 = e^{\frac{\pi}{2}}, a_4 = i.$$

So we can write

$$a_n = \begin{cases} e^{-\frac{\pi}{2}}, & \text{if } n \equiv 1 \mod 4, \\ -i & \text{if } n \equiv 2 \mod 4, \\ e^{\frac{\pi}{2}} & \text{if } n \equiv 3 \mod 4, \\ i & \text{if } n \equiv 0 \mod 4. \end{cases}$$