Bilkent University

Homework # 02
Math 503   Complex Analysis I
Due: 25 October 2020
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Solution Key

Q-1) Let $u(t)$, $v(t)$ be real valued continuous functions on the interval $[a,b]$. Let $K = \alpha + i\beta$, where α, β are some real numbers.

I: Show that
$$K\int_a^b f(t)\,dt = \int_a^b Kf(t)\,dt.$$

II: Show that
$$\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)\,dt|\,.$$

Answer:

I: If $K = i$, the result follows immediately from the definition of complex integrals. In general
$$\begin{aligned} K\int_a^b f(t)\,dt &= (\alpha+i\beta)\left(\int_a^b u(t)\,dt + i\int_a^b v(t)\,dt\right) \\ &= \int_a^b (\alpha u(t) - \beta v(t))\,dt + i\int_a^b (\alpha v(t) + \beta u(t))\,dt \\ &= \int_a^b \left[(\alpha u(t) - \beta v(t)) + i(\alpha v(t) + \beta u(t))\right]\,dt \\ &= \int_a^b Kf(t)\,dt \end{aligned}$$

II: Let
$$\int_a^b f(t)\,dt = re^{i\theta},$$
for some $r > 0$ and $\theta \in [0, 2\pi)$. Note the trivial fact that $r = e^{-i\theta}(re^{i\theta})$. Then we have
$$\begin{aligned} \left|\int_a^b f(t)\,dt\right| &= r = e^{-i\theta}\int_a^b f(t)\,dt \\ &= \int_a^b e^{-i\theta}f(t)\,dt \\ &= \int_a^b \mathrm{Re}(e^{-i\theta}f(t))\,dt \quad \textit{since the integrals are real} \\ &\leq \int_a^b \left|\mathrm{Re}(e^{-i\theta}f(t))\right|\,dt \\ &\leq \int_a^b \left|e^{-i\theta}f(t)\right|\,dt \\ &= \int_a^b |f(t)|\,dt. \end{aligned}$$

Q-2) Show by using only the definition of complex integrals that $\int_\gamma \frac{1}{z}\, dz = 2\pi i$, where γ is the unit circle centered at the origin and taken in the counterclockwise direction.

Answer:

On this circle we have $z = e^{i\theta}$, so $dz = ie^{i\theta}\, d\theta$. Then

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta}\, d\theta}{e^{i\theta}} = \int_0^{2\pi} i\, d\theta = 2\pi i.$$