Due Date: 11 March 2013, Monday NAME:..

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Math 504 Complex Analysis II – Take-Home Exam 01 – Solutions

1	2	3	4	5	TOTAL
25	25	25	25	0	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail.

For each question I will post the best student solution on the web. If there are more than one interesting solutions, I will post them all. Having your solution posted on the web will get you extra 10 points for each solution posted. These will be added to your total exam grades before an average is taken at the end of the semester.

NAME: STUDENT NO:

Q-1) Show that complex conjugation of Σ corresponds to reflection of S^2 in the plane $x_2 = 0$. What transformation of Σ correspond to the reflections in the plane $x_1 = 0$ and $x_3 = 0$? Show that the antipodal map $Q \mapsto -Q$ of S^2 is the composition of the above three reflections in any order and hence express it as a transformation of Σ.
[page 15, Exercise 1E]

Solution:

Let $z = x + iy \in \mathbb{C}$, $(x_1, x_2, x_3) \in \mathbb{R}^3$, and let $S^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin. Let $\pi : S^2 \to \mathbb{C}$ be the stereographic projection from the north pole $N = (0, 0, 1)$.

We have

$$\pi(x_1, x_2, x_3) = \frac{x_1}{1 - x_3} + i\frac{x_2}{1 - x_3},$$

and

$$\pi^{-1}(x + iy) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1x^2 + y^2 + 1} \right).$$

Let $R_i : S^2 \to S^2$ be the reflection around the plane $x_i = 0$, $i = 1, 2, 3$. Then we have

$$\begin{aligned} R_1(x_1, x_2, x_3) &= (-x_1, x_2, x_3), \\ R_2(x_1, x_2, x_3) &= (x_1, -x_2, x_3), \\ R_3(x_1, x_2, x_3) &= (x_1, x_2, -x_3). \end{aligned}$$

We check that
$\pi \circ R_1 \circ \pi^{-1}(x + iy) = -x + iy$, so $z \mapsto -\bar{z}$.

$\pi \circ R_2 \circ \pi^{-1}(x + iy) = x - iy$, so $z \mapsto \bar{z}$.

$\pi \circ R_3 \circ \pi^{-1}(x + iy) = \dfrac{x}{x^2 + y^2} + i\dfrac{y}{x^2 + y^2}$, so $z \mapsto 1/\bar{z}$.

Now let $R(x_1, x_2, x_3) = (-x_1, -x_2, -x_3)$ be the antipodal map on S^2. Then clearly R is the composition of R_1, R_2, R_3 in any order. We then see that

$$\pi \circ R \circ \pi^{-1}(z) = -\frac{1}{\bar{z}}.$$

NAME: STUDENT NO:

Q-2) Let f be a rational function whose poles in $\mathbb{C}$ are $\beta_1, \ldots, \beta_q$. Prove that there exists unique polynomials $\phi_0, \ldots, \phi_q$ with zero constant term such that

$$f(z) = \phi_0(z) + \sum_{i=1}^{q} \phi_i \left(\frac{1}{z - \beta_i} \right) + \text{constant}.$$

Illustrate this result with reference to the function

$$f(z) = \frac{z^2}{(z-1)^2(z-2)}.$$

[page 16, Exercise 1L]

Solution:

Any rational function $f(z)$ can be written as

$$f(z) = G(z) + \frac{P(z)}{Q(z)},$$

where G, P, Q are polynomials with $\deg P(z) < \deg Q(z)$.

Set $\phi_0(z) = G(z) - G(0)$.

Let the order of the pole β_i of $f(z)$ be $r_i > 0$ and let $\dfrac{c(i, r_i)}{(z-\beta_i)^{r_i}} + \dfrac{c(i, r_i - 1)}{(z-\beta_i)^{r_i - 1}} + \cdots + \dfrac{c(i, 1)}{z - \beta_i}$ be the principal part of $f(z)$ at β_i, where $c(i, r_i) \neq 0, c(i, r_i - 1), \ldots, c(i, 1)$ are complex constants, $i = 1, \ldots, q$.

Now let $\phi_i(z) = c(i, r_i) z^{r_i} + \cdots + c(i, 1) z$, $i = 1, \ldots, q$.

It is now clear that

$$f(z) = \phi_0(z) + \sum_{i=1}^{q} \phi_i \left(\frac{1}{z - \beta_i} \right) + \text{constant},$$

since the difference between the right hand side and the left hand side is a function which has no poles in $\Sigma = \mathbb{C} \cup \{\infty\}$. Hence the difference is a constant.

Applying this game to the function $f(z) = \frac{z^2}{(z-1)^2(z-2)}$, we find that

$$f(z) = \frac{4}{z-2} - \frac{3}{(z-1)^2} - \frac{3}{z-1}.$$

NAME: STUDENT NO:

Q-3) Let $f(z)$ be a rational function such that $|z| = 1$ implies $|f(z)| = 1$. Show that α is a zero of $f(z)$ if and only if $1/\bar{\alpha}$ is a pole of $f(z)$, and hence find the most general form of $f(z)$.
[page 16, Exercise 1M]

Solution:

If $z = 0$ is a zero of $f(z)$, then for some positive integer k we have $f(z) = z^k g(z)$ for some rational function $g(z)$ which satisfies $g(e^{i\theta}) = 1$ and $g(0) \neq 0$. Similarly $f(z) = g(z)/z^k$ if the origin is a pole of f.

So we may assume now that the origin is neither a zero nor a pole of f.

If f has no zeros or poles inside D, then using the maximum modulus principle for both $f(z)$ and $1/f(z)$ inside D, we find that $|f(z)| = 1$ for $z \in D$, which contradicts the open mapping property of a non-constant analytic function. So f must have a zero or a pole in D unless f is constant.

We now want to classify all non-constant rational functions f which satisfy the property that $|f(e^{i\theta})| = 1$, is analytic on D, and having finitely many zeros $\alpha_1, \ldots, \alpha_n$ and finitely many poles $\beta_1, \ldots, \beta_m$ in D, where $n + m > 0$.

For any $\alpha \in D$, define a rational function

$$B(z, \alpha) = \frac{z - \alpha}{1 - \bar{\alpha} z}.$$

Observe that $|B(e^{i\theta}, \alpha)| = 1$.

Consider the function

$$H(z) = f(z)\,\frac{\prod_{j=1}^{n} B(z, \beta_j)}{\prod_{j=1}^{n} B(z, \alpha_j)}.$$

Now, H is a rational function satisfying $|H(z)| = 1$ for $|z| = 1$, and has no zeros or poles inside D. From the above arguments we conclude that $H(z) = \lambda$ for some constant λ with $|\lambda| = 1$.

Finally, if f is a non-constant rational function such that $|z| = 1$ implies $|f(z)| = 1$, then the general form of f is

$$f(z) = \lambda z^k \prod_{j=1}^{n} B(z, \alpha_j),$$

where $|\lambda| = 1$, k is an integer, $\alpha_1, \ldots, \alpha_n \in \mathbb{C} \setminus \{0\}$, with $|k| + n > 0$.

Here is another idea. For any rational function $f(z)$ define another rational function of z as

$$f^*(z) := \frac{1}{\overline{f(\frac{1}{\bar{z}})}}.$$

In our case, for $z \in \partial D$ we have both $f(z)\overline{f(z)} = 1$ and $z\bar{z} = 1$. Putting these together we find that

$$f(z) = f^*(z)$$

for all $z \in D$. Two rational functions agreeing on ∂D must be identical on $\mathbb{C}$, so we have $f(z) = f^*(z)$ for all $z \in \mathbb{C}$.

Now it is immediate to see that α is a pole of f if and only if $1/\bar{\alpha}$ is a pole of f^* which is f. The rest now follows as above.

NAME: STUDENT NO:

Q-4) Investigate the covering of the sphere by the sphere associated with the rational function

$$f(z) = \frac{z^3}{z^4 + 27}.$$

[page 16, Exercise 1N]

Solution:

The poles of f are simple so they don't give branch points. Also $f(1/z) = 0$ has a simple zero at $z = 0$, so ∞ is not a branch point either.

We then look at the zeros of $f'(z) = \dfrac{z^2(81 - z^4)}{(z^4 + 27)^2}$, whose roots are $0, \pm 3, \pm 3i$. We examine the branching at these points separately.

Note that $f''(z) = \dfrac{2z\,(z^8 - 324\,z^4 + 2187)}{(z^4 + 27)^3}$ and $f'''(z) = \dfrac{6z^{12} - 837\,z^8 + 22599\,z^4 - 19683}{(z^4 + 27)^4}$

We check that $f''(0) = 0$ but $f'''(0) \neq 0$, so $z = 0$ is a branch point of order 2, i.e. three sheets meet at $z = 0$.

For the other critical points the second derivative does not vanish, so each is a branch point of order 1, i.e. at each of these points two sheets come together.