Due Date: 20 March 2013, Wednesday NAME:..

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Math 504 Complex Analysis II – Take-Home Exam 03 – Solutions

1	2	3	4	5	TOTAL
25	25	25	25	0	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail.

For each question I will post the best student solution on the web. If there are more than one interesting solutions, I will post them all. Having your solution posted on the web will get you extra 10 points for each solution posted. These will be added to your total exam grades before an average is taken at the end of the semester.

NAME: STUDENT NO:

Q-1)

(i) Show that $\mathbb{Z}[i]$ and $\mathbb{Z}[\rho]$ are discrete subgroups of $\mathbb{C}$, where $\rho = \frac{1}{2}(1+\sqrt{-3})$.

(ii) Show that $\mathbb{Z}[\sqrt{2}]$ is not a discrete subgroup of $\mathbb{C}$.

[page 120, Exercise 1A]

Solution:

(i) Let $p = m + in \in \mathbb{Z}[i]$. If $p \neq 0$, then $|p|^2 = m^2 + n^2 \geq 1$. Let U be the unit open disc in $\mathbb{C}$ centered at the origin. Then $p + U$ is an open neighborhood of p containing no other element of $\mathbb{Z}[i]$.

Similarly if $p = m + \rho n \in \mathbb{Z}[\rho]$, then $|p|^2 = (m + \dfrac{n}{2})^2 + \dfrac{3}{4}n^2 \geq \dfrac{3}{4}$. This shows, as above, that $\mathbb{Z}[\rho]$ is discrete.

(ii) We claim that for any $\epsilon > 0$ there exists an element $p \in \mathbb{Z}[\sqrt{2}]$ with $|p| < \epsilon$. Assume to the contrary that $p_0 = m_0 + n_0\sqrt{2}$ is the smallest nonzero element in $p \in \mathbb{Z}[\sqrt{2}]$. For any non-zero $p = u + v\sqrt{2}$ let k be the smallest integer that makes $|p - p_0|$ minimum. Since $|p_0|$ is smallest non-zero element, we must have $p - p_0 = 0$, but this implies that $\sqrt{2} = \dfrac{m_0 - u}{v - n_0}$ which is a contradiction. Hence $p \in \mathbb{Z}[\sqrt{2}]$ is not discrete.

(Another idea) Since $0 < -1 + \sqrt{2} < 1$, for any $\epsilon > 0$ there exists an integer $N > 0$ such that $(-1+\sqrt{2})^N < \epsilon$. So $p \in \mathbb{Z}[\sqrt{2}]$ is not discrete.

NAME: STUDENT NO:

Q-2) Find conditions on the integers a and b such that $a\omega_1 + b\omega_2$ is a basis element of the lattice generated by ω_1 and ω_2.
[page 120, Exercise 3E]

Solution:

Let $c\omega_1 + d\omega_2$ be the other element of the basis. By Theorem 3.4.2 on page 65 we must have $ad - bc = \pm 1$. This is equivalent to saying that $(a, b) = 1$.

NAME: STUDENT NO:

Q-3) Let Ω be a lattice. For which λ, the map $\omega \mapsto \lambda\omega$ is an automorphism of Ω?
[page 120, Exercise 3G]

Solution:

Let $\omega \in \Omega$ be a non-zero element of smallest length. Then $|\lambda\omega| \geq |\omega|$, so $|\lambda| \geq 1$. But since $1/\lambda : \Omega \to \Omega$ is also an automorphism, for the same reasoning we must have $|1/\lambda| \geq 1$. This forces $|\lambda| = 1$.

Since λ induces an automorphism, if $\{\omega_1, \omega_2\}$ is a basis, then $\{\lambda\omega_1, \lambda\omega_2\}$ is also a basis. Hence there is a bases change matrix which represents the automorphism induced by λ. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \lambda\omega_1 \\ \lambda\omega_2 \end{pmatrix} = \lambda \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

where $a, b, c, d \imath \mathbb{Z}$ and $ad - bc = \pm 1$.

Since λ now occurs as an eigenvalue, it must satisfy the equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

Since $|\lambda| = 1$, this is an elliptic transformation, so $|a + d| \leq 1$.

Now we examine the possible cases.

$ad - bc = 1$ and $a + d = 0$.
In this case we have $\lambda^2 + 1 = 0$, so $\lambda = \pm i$.

$ad - bc = 1$ and $a + d = 1$.
In this case we have $\lambda^2 - \lambda + 1 = 0$, so $\lambda = \dfrac{1 \pm i\sqrt{3}}{2}$.

$ad - bc = 1$ and $a + d = -1$.
In this case we have $\lambda^2 + \lambda + 1 = 0$, so $\lambda = \dfrac{-1 \pm i\sqrt{3}}{2}$.

$ad - bc = -1$ case.
In this case we have

$$\left(\lambda - \frac{a+d}{2}\right) = \left(\frac{a+d}{2}\right)^2 + 1.$$

This forces λ to be real but then $|\lambda| = 1$ forces $\lambda = \pm 1$.

Hence the only possible values of an automorphism inducing λ are

$$\pm 1, \quad \pm i, \quad \frac{1 \pm i\sqrt{3}}{2}, \quad \frac{-1 \pm i\sqrt{3}}{2}.$$

NAME: STUDENT NO:

Q-4)

(i) Show that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ converges normally on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$.

(ii) Show that $\displaystyle\pi^2 \operatorname{cosec}^2 \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

[page 121, Exercise 3I and 3J]

Solution:

Let K be a compact subset of the complex plane. Then there exists an index N such that for all $n \geq N$, the functions $(z-n)^{-2}$ are analytic on K.

Note that for all $n \geq N$, and all $z \in K$, we have $|z * n| > 0$ and also

$$\lim_{n\to\infty} \frac{\frac{1}{|z-n|^2}}{\frac{1}{n^2}} = 1,$$

so by the limit comparison test

$$\sum_{n=N}^{\infty} \frac{1}{|n-z|^2}$$

converges. This proves that $\displaystyle\sum_{n=N}^{\infty} \frac{1}{(n-z)^2}$ converges normally, and hence uniformly on compact subsets of $\mathbb{C}$

For the second part we follow the hints in the book. Let

$$h(z) = \pi^2 \operatorname{cosec}^2 \pi z - \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

Clearly $h(z+1) = h(z)$, so h is periodic with period 1.

Laurent series of $\pi^2 \operatorname{cosec}^2 \pi z$ at $z = 0$ is

$$\frac{1}{z^2} + \frac{\pi^2}{3} + \frac{\pi^4}{15} + z^2 + \cdots.$$

Thus we have

$$h(z) = \frac{1}{z^2} + \frac{\pi^2}{3} + \frac{\pi^4}{15} + z^2 + \cdots - \frac{1}{z^2} - \sum_{\substack{n=-\infty \\ n\neq 0}} \frac{1}{(z-n)^2}.$$

This shows that $h(0)$ is well defined after canceling out $1/z^2$.

By periodicity $h(n)$ is defined at all integers. Thus h is continuous and restricting z to the closed and bounded real interval $[0,1]$, we see that $|h(z)| \leq M$ for some $M > 0$. Again by periodicity we have $|h(z)| < M$ for all real z.

We now recall some trigonometric identities.

$$\sin^2 \frac{\pi z}{2} = \frac{1-\cos \pi z}{2}, \text{ and } \sin^2 \frac{\pi(z+1)}{2} = \cos^2 \frac{\pi z}{2} = \frac{1+\cos \pi z}{2}.$$

Using these we have

$$\begin{aligned} h(\frac{z}{2}) + h(\frac{z+1}{2}) &= \frac{\pi^2}{\sin^2 \frac{\pi z}{2}} - \sum_{n=-\infty}^{\infty} \frac{4}{(z-2n)^2} + \frac{\pi^2}{\cos^2 \frac{\pi z}{2}} - \sum_{n=-\infty}^{\infty} \frac{4}{(z-(2n-1))^2} \\ &= \frac{2\pi^2}{1-\cos \pi z} + \frac{2\pi^2}{1+\cos \pi z} - \sum_{n=-\infty}^{\infty} \frac{4}{(z-n)^2} \\ &= \frac{4\pi^2}{1-\cos^2 \pi z} - \sum_{n=-\infty}^{\infty} \frac{4}{(z-n)^2} \\ &= \frac{4\pi^2}{\sin^2 \pi z} - \sum_{n=-\infty}^{\infty} \frac{4}{(z-n)^2} \\ &= 4h(z). \end{aligned}$$

We showed that

$$h(\frac{z}{2}) + h(\frac{z+1}{2}) = 4h(z),$$

for all $z \in \mathbb{C}$. Restricting z to be real we find that

$$|4h(z)| \le |h(\frac{z}{2})| + |h(\frac{z+1}{2})| < 2M,$$

which implies that for all real z,

$$|h(z) < M/2.$$

Iterating the above arguments we find that for all real z,

$$|h(z)| < M/2^n$$

for all positive integers n. This shows that $h(z) = 0$ for all real z. But then this forces the entire function $h(z)$ to be identically zero, which is precisely what we want to show.

Another idea is to start with the infinite product expression

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right).$$

Let

$$f(z) = \frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right).$$

Calculating the logarithmic derivative of f we find

$$g(z) = \frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n}\right).$$

Differentiating term by term we find

$$g'(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

Next noting that $f(z) = \dfrac{\sin \pi z}{\pi}$, calculate $g'(z)$ directly to find

$$g'(z) = \left(\frac{f'(z)}{f(z)} \right)' = \pi^2 \operatorname{cosec}^2 \pi z.$$