Chapter 1

Affine Varieties

1.1 First Definitions

Affine Space: We fix an algebraically closed field k. The affine n space over k is denoted by $\mathbb{A}^n_k$ which is simply the set of n-tuples of elements of k. If k is understood it can also be denoted simply by $\mathbb{A}^n$. It differs from k^n by the absence of a k-vector space structure.

Zero Set: For any ideal J of the polynomial ring $k[x_1, \ldots, x_n]$ we denote by $Z(J)$ the common zeros of all polynomials in J,

$$Z(J) = \{p \in \mathbb{A}^n \mid f(p) = 0 \text{ for all } f \in J\}.$$

Every ideal in $k[x_1, \ldots, x_n]$ is finitely generated. If $J = (f_1, \ldots, f_r)$, then we denote $Z(J)$ also by $Z(f_1, \ldots, f_r)$.

Definition 1 *A subset X of $\mathbb{A}^n$ is called an algebraic set if it is of the form $X = Z(J)$ for some ideal $J \subset k[x_1, \ldots, x_n]$.*

Some immediate examples of algebraic sets are $\emptyset = Z(1)$, $\{(a_1, \ldots, a_n)\} = Z(x_1 - a_1, \ldots, x_n - a_n)$ and $\mathbb{A}^n = Z(0)$. If $k = \mathbb{C}$, then a nonempty proper subset of $\mathbb{A}^n$ which is open with respect to the usual metric topology is not an algebraic set.

Zariski Topology: We put a new topology on $\mathbb{A}^n$ by declaring that the collection of closed sets will consist only of algebraic sets. The topology thus defined is called the *Zariski topology.*

Affine Variety: A closed set in a topological space is called *irreducible* if it is not the union of two proper nonempty closed subsets. An affine algebraic set is called an *affine variety* if it is irreducible in the Zariski topology. We may drop the mention of "affine" when it is clear from the context, for example when we are working in $\mathbb{A}^n$ or working with the polynomial ring $k[x_1, \ldots, x_n]$.

If $f \in k[x_1, \ldots, x_n]$ is an irreducible polynomial, then $Z(f)$ is a variety. $Z(x_1x_2)$ is not a variety since it is the union of the closed sets $Z(x_1)$ and $Z(x_2)$.

Dimension: The *dimension* of a closed set V in a topological space is defined to be the maximal integer m, if it exists, for which we have a chain of inclusions

$$V = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_m$$

where each V_i is a nonempty closed subset. If no such m exists, the dimension of V is defined to be infinite. The dimension of an open set is the dimension of its closure.

The *dimension of an algebraic variety* is defined as its dimension as a closed set in the Zariski topology. The dimension of an algebraic set is defined to be the largest of the dimensions of its irreducible components. Varieties of dimension one and two are called curves and surfaces, respectively. An n dimensional variety is generally referred a to as an n-fold when $n > 2$.

For any nonconstant $f \in k[x_1, \ldots, x_n]$, the algebraic set $Z(f)$ is called a *hypersurface*, and if f is linear it is called a *hyperplane.* The dimension of a hypersurface in $\mathbb{A}^n$ is $n - 1$.

Ideal of a Set: Starting with an arbitrary subset X of $\mathbb{A}^n$ define the set of all polynomials which vanish on X by

$$I(X) = \{f \in k[x_1, \ldots, x_n] \mid f(p) = 0 \text{ for all } p \in X \}.$$

This is a radical ideal in $k[x_1, \ldots, x_n]$.

We have the immediate relations;
(i) $X \subset Z(I(X))$ for any $X \subset \mathbb{A}^n$ and
(ii) $J \subset I(Z(J))$ for any ideal $J \subset k[x_1, \ldots, x_n]$.

Equality in (i) holds if X is already an algebraic set. Otherwise $Z(I(X))$ is simply the closure $\bar{X}$ of X.

To understand when the equality holds in (ii) we need the following theorem.

Theorem 2 (Hilbert's zero theorem) *If J is any ideal in $k[x_1, \ldots, x_n]$, then $I(Z(J))$ is $\sqrt{J}$, the radical of J.*

Proof: See Eisenbud [1]. □

This theorem fails when k is not algebraically closed.

We have an inclusion reversing correspondence between radical ideals and algebraic sets. In other words if $X \subset Y \subset \mathbb{A}^n$, then $I(Y) \subset I(X)$. And if $I_1 \subset I_2$ are two ideals in $k[x_1, \ldots, x_n]$, then $Z(I_2) \subset Z(I_1)$.

The algebraic set $Z(J)$ is irreducible if and only if J is a prime ideal. Similarly $I(X)$ is a prime ideal if and only if X is an algebraic variety.

The dimension of an algebraic variety X is equal to the (Krull) dimension of the prime ideal $I(X)$ in $k[x_1, \ldots, x_n]$.

Definition 3 *For an ideal J in $k[x_1, \ldots, x_n]$ we define $\mu(J)$ to be the number of elements in a minimal generating set for J. If X is an algebraic set we define $\mu(X)$ as the minimum integer r such that there exist r polynomials $f_1, \ldots, f_r$ with $X = Z(f_1, \ldots, f_r)$ We define* $\operatorname{codim}(X)$*, the codimension of X, as $n - \dim(X)$, where the dimension of X is its dimension in the Zariski topology of $\mathbb{A}^n$*

We immediately have the inequalities $0 \leq \operatorname{codim}(X) \leq \mu(X) \leq \mu(J(X))$.

In $\mathbb{A}^2$, when C is a curve, we always have $1 = \operatorname{codim} C = \mu(C) = \mu(I(C))$.

Every codimension one variety X in $\mathbb{A}^n$ is a hypersurface and necessarily $n - 1 = \mu(X) = \mu(I(X))$, see [3].

For every variety X in $\mathbb{A}^n$, it is known that $\mu(X) \leq n$, see [2]. However for any given integer m, there exists a variety X in $\mathbb{A}^n$ with $\mu(I(X)) \geq m$. One such variety will be discussed in the next section.

Complete Intersections: For any variety X, if $\mu(X) = \operatorname{codim} X$, then X is called a *set theoretical complete intersection*, STCI for short. If further $\mu(I(X)) = \operatorname{codim} X$, then X is an *ideal theoretical complete intersection*, or ITCI. We have examples of STCI varieties which are not ICTI.

Conjecture: It is conjectured that all curves in $\mathbb{A}^3$ are STCI.

Exercises

1. Show that affine algebraic sets satisfy the axioms for the closed sets of a topology, i.e. show that the intersection of an arbitrary collection of algebraic sets is algebraic and the union of two algebraic sets is algebraic.

2. Show that an arbitrary union of algebraic sets need not be algebraic.

3. Show that the codimension of a hypersurface is one.

4. Show that $I(X)$ for any set $X \subset \mathbb{A}^n$ is a radical ideal.

5. Give an example of an ideal J in $\mathbb{R}[x_1, \ldots, x_n]$ for which $I(Z(J))$ is not the radical of J.

6. For any ideal J in $k[x_1, \ldots, x_n]$, show that the algebraic set $Z(J)$ is irreducible if and only if J is a prime ideal.

7. For any ideal J in $k[x_1, \ldots, x_n]$, show that $Z(J)$ is singleton if J is a maximal ideal. Is the converse true?