Due on January 9, 2007, Tuesday.

MATH 591 Take-Home Final Exam

1: Let $\{U_i\}$ be the usual open covering of $\mathbb{P}^r$ and let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Show that the Čech cohomology with coefficients in $\mathcal{F}$ calculated using this covering gives the cohomology of $\mathbb{P}^r$. Conclude that $H^n(\mathbb{P}^r, \mathcal{F}) = 0$ for $n > r$.

2: Let $\mathbb{P}^r$ be the projective space, $\{U_i\}$ the usual covering and $[x_0 : \cdots : x_n]$ the homogeneous coordinates. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ with $\mathcal{F}_i = \mathcal{F}(U_i)$. Fix an integer n. Define a map $\theta_{ij} : \mathcal{F}_j(U_i \cap U_j) \to \mathcal{F}_i(U_i \cap U_j)$ by $\theta_{ij}(f_j) = (x_j^n/x_i^n)f_j$. This is clearly an isomorphism. Gluing the sheaves $\mathcal{F}_i$ with this isomorphism we obtain a new sheaf which we denote by $\mathcal{F}(n)$. Show that $\mathcal{F}(0) \cong \mathcal{F}$, $\mathcal{F}(n)(m) \cong \mathcal{F}(n+m)$. Show that the elements of $\mathcal{O}(n)_x$, where $\mathcal{O}$ is the structure sheaf and $x \in \mathbb{P}^r$, consist of rational functions of the form P/Q where P and Q are polynomials with $Q(x) \neq 0$ and $\deg P - \deg Q = n$.

3: Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$. Show that there exists an integer N_0, depending on $\mathcal{F}$, such that for all $n \geq N_0$ and for all $x \in \mathbb{P}^r$, the $\mathcal{O}_x$-module $\mathcal{F}(n)_x$ is generated by global sections of $\mathcal{F}(n)$.
Note: This can be found in Hartshorne's book on page 121 as Theorem 5.17 but the original proof in Serre FAC §55 is much easier to follow.