Bilkent University

Take-Home Exam # 03
Math 633 Algebraic Geometry
Due on: 5 December 2019 Thursday - Class Time
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Solution Key

Q-1) Hartshorne Exercise II.1.21 pages 68-69
Some Examples of Sheaves on Varieties. Let X be a variety over an algebraically closed field k, as in Ch. I. Let $\mathcal{O}_X$ be the ring of regular functions on X (1.0.1).

(a) Let Y be a closed subvariety of X. For each open set $U \subseteq X$, let $\mathcal{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathcal{I}_Y(U)$ is a sheaf. It is called the *sheaf of ideals* $\mathcal{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.

(b) Show that the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i : Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

(c) Now let $X = \mathbb{P}^1$, and let Y be the union of two distinct points $P, Q \in X$. Thus by (b) we have an exact sequence of sheaves on X

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0.$$

Show however that the induced map on the global sections $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, i_*\mathcal{O}_Y)$ is not surjective. This shows that the global section functor $\Gamma(X, \cdot)$ is not exact (cf. (Ex. 1.8) which shows that it is left exact).

(d) Again let $X = \mathbb{P}^1$, let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathcal{K}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathcal{K}$. Show that the quotient sheaf $\mathcal{K}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\sum_{P \in X} i_P(I_P)$, where I_P is the group $K/\mathcal{O}_P$, and $i_P(I_P)$ denotes the skyscraper sheaf (Ex. 1.17) given by I_P at the point P.

(e) Finally show that in the case of (d) the sequence

$$0 \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{K}/\mathcal{O}) \to 0$$

is exact. (This is an analogue of what is called the "first Cousin problem" in several complex variables. See Gunning and Rossi [1, p. 248].)

Solution:

(a)
Let U be an open subset of X covered by the open subsets V_i. Let $s \in \mathcal{I}_Y(U)$ be such that $s|_{V_i} = 0$ for all i. Then considering s as an element of $\mathcal{O}_X(U)$, we know that $s = 0$. Moreover if for each i we have an $s_i \in \mathcal{I}_Y(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all i and j, then there is an element in $\mathcal{O}_X(U)$ such that $s|_{V_i} = s_i$. Since each s_i vanishes on Y, s also vanishes on Y and is in $\mathcal{I}_Y(U)$. Hence $\mathcal{I}_Y$ is a sheaf.

(b)
At each point $p \in Y$, the map $(\mathcal{O}_X)_p \to (i_*(\mathcal{O}_Y))_p$ is the restriction map and is surjective, the kernel being $(\mathcal{I}_Y)_p$. Thus the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to the sheaf $\mathcal{O}_Y$.

(c)
Global regular functions on $\mathbb{P}^1$ are constants. Restriction of this to Y gives a function whose value at P and Q are the same. However we can clearly have functions on Y with different values at P and Q. Hence the global section functor is not exact.

(d)
Let $\mathcal{S}$ denote the sheaf $\sum_{p\in X} i_P(I_P)$. We construct a short sequence

$$0 \to \mathcal{O} \to \mathcal{K} \xrightarrow{\phi} \mathcal{S} \to 0$$

and show that on the germ level it is an exact sequence, which establishes the required isomorphism.

Let $q \in X$ and f a germ in the stalk $\mathcal{K}_q$. $\phi_q(f) = 0$ if and only if f is regular at q. Hence it remains to show that ϕ_q is surjective. An element of I_q is an equivalence class of elements of the form h/z^n for some positive integer n, where z is a local coordinate centered at q and h is a rational function of z regular at q. If $c_n = h(q)$, then there are a polynomial h_1, h_2 such that $h/z^n = c_n/z^n + h_1/(z^m h_2)$ for some $m < n$, where $h_2(q) \neq 0$. Proceeding inductively there are constants $c_n, \dots, c_1$ and a polynomial h_0 such that $h/z^n = c_n/z^n + \cdots + c_1/z + h_0/h_2$. But h_0/h_2 is regular at q, so h/z^n in I_q is in the same equivalence class of $g = c_n/z^n + \cdots + c_1/z$. This function g then represents an element of the stalk $\mathcal{K}_q$, and note that the only pole of g is at q. This shows that ϕ_q is surjective (we actually showed more than necessary but we will use this in (e)). This completes the proof that the above short sequence is exact and the required isomorphism now follows.

(e)
The global sections functor is left exact, see (Ex. 1.8). So it suffices to show surjectivity on the right.

Let $f \in \Gamma(X, \mathcal{K}/\mathcal{O})$. From part (d) we can consider f as an element of $\Gamma(X, \sum_{p\in X} i_P(I_P))$. $f = f_1 + \cdots + f_r$ where each f_i is equivalent to a principal part of the form $c_n/z^n + \cdots + c_1/z$, as in (d), where z is a local coordinate centered at some q_i. For each such principal part there is a rational function g_i as in (d) which is regular everywhere except at p_i and differs from f by a regular function at p_i. Let $g = g_1 + \cdots + g_r$. Then g is an element of $\Gamma(X, \mathcal{K})$ which maps to f, giving the right exactness.

This is a Cousin problem in the sense that given a finite number of points in $\mathbb{P}^1$ and principal parts at those points, then there exists a rational function which is regular everywhere except the given points and has precisely the assigned principal parts at those points.