Bilkent University

Take-Home Exam # 06
Math 633 Algebraic Geometry
Due on: 31 December 2019 Thuesday
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Solution Key

Q-1) Explain in detail the calculation of $H^1(\mathbb{C}^2 \setminus \{0\}, \mathcal{O})$
This is *Computation* 4 in Griffiths & Harris, *Principles of Algebraic Geometry*, on page 49, where the main lines of the argument are listed. Fill in the missing details.

Solution:

By Hartog's theorem (p. 7) we have $\mathcal{O}(\mathbb{C}^2 \setminus \{0\}) = \mathcal{O}(\mathbb{C}^2)$.

Let
$$U_1 = \{(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\} \mid z_1 \neq 0\} \text{ and } U_2 = \{(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\} \mid z_2 \neq 0\}.$$
Then $U = \{U_1, U_2\}$ is an open covering of $\mathbb{C}^2 \setminus \{0\}$.

Notice that with respect to this covering $C^2(U, \mathcal{O}) = 0$ by definition, since there are no three distinct open sets in the cover. Therefore the first cohomology will be calculated from the following sequence.

$$C^0(U, \mathcal{O}) \xrightarrow{\delta} C^1(U, \mathcal{O}) \to 0,$$

where δ is the coboundary map which we now define.

A cochain $f \in C^0(U, \mathcal{O})$ is of the form $f = (f_1, f_2)$ where $f_j \in \mathcal{O}(U_j)$, $j = 1, 2$.

We notice that $1/z_1$ is a function in $\mathcal{O}(U_1)$ but not in $\mathcal{O}(U_2)$. Similarly $1/z_2$ is a function in $\mathcal{O}(U_2)$ but not in $\mathcal{O}(U_1)$. Therefore f_1 and f_2 have the following Laurent expansions.

$$f_1(z_1, z_2) = \sum_{\substack{m > -\infty \\ n \geq 0}} c^{(1)}_{mn} z_1^m z_2^n, \quad \text{and} \quad f_2(z_1, z_2) = \sum_{\substack{m \geq 0 \\ n > -\infty}} c^{(2)}_{mn} z_1^m z_2^n,$$

where $c^{(1)}_{mn}, c^{(2)}_{mn}$ are complex constants. We then have

$$\delta f = f_2\Big|_{U_1 \cap U_2} - f_1\Big|_{U_1 \cap U_2}.$$

On the other hand an element $g \in C^1(U, \mathcal{O})$ is of the form $g = (g_{12})$ where $g_{12} \in \mathcal{O}(U_1 \cap U_2) = \mathcal{O}(\mathbb{C}^* \times \mathbb{C}^*)$. Hence such a g_{12} has a Laurent expansion of the form

$$g_{12}(z_1, z_2) = \sum_{\substack{m > -\infty \\ n > -\infty}} c_{mn} z_1^m z_2^n,$$

where c_{mn} are complex constants. This shows that $\delta C^0(U, \mathcal{O})$ does not cover all of $C^1(U, \mathcal{O})$ since Laurent series which contain terms of the form $z_1^m z_2^n$ where both m and n are negative are not in the image of δ.

Therefore we found that

$$\check{H}^1(U,\mathcal{O}) \cong \mathbb{C}[z_1^{-1}, z_2^{-1}] \setminus \mathbb{C}.$$

Now we observe that $H^1(U_j,\mathcal{O}) = 0$, $j = 1, 2$, (p. 46), so U is a Leray covering of 1st order. Hence

$$\check{H}^1(\mathbb{C}^2 \setminus \{0\},\mathcal{O}) \cong \check{H}^1(U,\mathcal{O}) \cong \mathbb{C}[z_1^{-1}, z_2^{-1}] \setminus \mathbb{C}.$$

The version of Leray theorem we have just used is the following.

Theorem: Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X such that $\check{H}^1(U_i,\mathcal{F}) = 0$ for every $i \in I$. Then $\check{H}^1(X,\mathcal{F}) \cong \check{H}^1(\mathcal{U},\mathcal{F})$.

Proof: This is Theorem 12.8 on page 101 in *Lectures on Riemann Surfaces* by Otto Forster, Springer-Verlag (1981).