Cohomological Study of Weighted Projective Spaces

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Introduction

Let $\pi : E \longrightarrow X$ be a complex vector bundle of constant rank and with a compact base X. Assume given an operation $\sigma : \mathbb{C}^* \times E \longrightarrow E$ of the multiplicative group $\mathbb{C}^*$ on E, compatible with the natural operation of $\mathbb{C}^*$. In the algebraic (resp. analytic) situation, σ is supposed to be defined by polynomial (resp. holomorphic) functions, so that E splits into a direct sum of vector bundles $E = E_0 \oplus \cdots \oplus E_n$ (over X), such that the operation σ is determined by the characters of $\mathbb{C}^*$ in the following way

$$\sigma(\lambda, (u_0, \cdots, u_n)) = (\chi_0(\lambda)\, u_0, \cdots, \chi_n(\lambda)\, u_n) \quad (u_i \in E_i)\ ,$$

where χ_i are characters of $\mathbb{C}^*$. These have the form $\chi_i(\lambda) = \lambda^{q_i}$ $(\lambda \in \mathbb{C}^*, q_i \in \mathbb{Z})$. Consider the case $q_0 > 0, \cdots, q_n > 0$, and denote by

$$\widetilde{\mathbb{P}}(E) = \mathbb{P}(\bigoplus_0^n E_i; q_0, \cdots, q_n) \quad \text{(with projection } \rho \text{ on } X\ )$$

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the (topological) quotient of $E^*= E\backslash X$ modulo the operation σ. The problem we are interested in is the computation, à la GROTHENDIECK, of the integral cohomology $H^*(\widetilde{\mathbb{P}}(E))$ (étale cohomology in the algebraic case). More precisely we want to describe:
1) the additive structure of $H^*(\widetilde{\mathbb{P}}(E))$, as an $H^*(X)$-module (via ρ^*). The result is that $H^*(\widetilde{\mathbb{P}}(E))$ is a finitely generated free $H^*(X)$-module, and an explicit basis can be constructed.
2) the multiplicative structure of $H^*(\widetilde{\mathbb{P}}(E))$. Let us recall that in the standard case of the projective bundle $\mathbb{P}(E)$ (i.e. $n = 0$ or $q_0 = \cdots = q_n = 1$), the multiplicative structure of $H^*(\widetilde{\mathbb{P}}(E))$ is given by a unique relation

$$\xi_E^{r+1} = -\sum_1^{r+1} c_i\, \xi_E^{r-i+1} \qquad \text{(see III.3)}$$

which, at the same time, defines, in an algebraic way, the CHERN classes of the vector bundle E (after GROTHENDIECK [G1]). So the question is to find a generalization of the preceding relation for the weighted projective bundle $\widetilde{\mathbb{P}}(E)$, and to see how much it determines the multiplicative structure of $H^*(\widetilde{\mathbb{P}}(E))$. Instead of the usual CHERN classes $c_i(E)$ of E, we encounter here the following "characteristic" classes (given by their total class)

$$\tilde{c}(E) = (\psi^{l/q_0} c(E_0)) \cdots (\psi^{l/q_n} c(E_n))$$

where $l = \operatorname{lcm}\{q_0, \cdots, q_n\}$, ψ^i is the i-th ADAMS operation and $c(E_j)$ is the total CHERN class of E_j. These "twisted" classes have the same properties as CHERN classes, except the additivity one which is replaced by the following. Assume given $E' = \bigoplus_0^{n'} E'_i$ with weights $q'_0, \cdots, q'_{n'}$ and $E'' = \bigoplus_0^{n''} E''_i$ with weights $q''_0, \cdots, q''_{n''}$ (two copies of the situation above). Put $l' = \operatorname{lcm}\{q'_0, \cdots, q'_{n'}\}$, $l'' = \operatorname{lcm}\{q''_0, \cdots, q''_{n''}\}$ and $m = \operatorname{lcm}\{l', l''\}$. Weighting $E' \oplus E''$ by q'_i, q''_j in the obvious way, we obtain

$$\tilde{c}\,(E' \oplus E'') = (\psi^{m/l'} \tilde{c}(E'))\, (\psi^{m/l''} \tilde{c}(E''))\,.$$

We shall say here nothing more about the multiplication in $H^*(\widetilde{\mathbb{P}}(E))$.

This, and other more explicit results, will be the subject for a forthcoming work [A1].

To construct a basis of the free $H^*(X)$-module $H^*(\widetilde{\mathbb{P}}(E))$, we first study (in details) the case when the space X is a point, i.e. $\widetilde{\mathbb{P}}(E)$ is a weighted projective space $\widetilde{\mathbb{P}}^r$, and then use the LERAY-HIRSCH theorem.

The integral cohomology $H^*(\widetilde{\mathbb{P}}^r)$ has been first computed by KAWASAKI [K]. Then the author determined the étale cohomology of the scheme $\widetilde{\mathbb{P}}^r$. For this he first recomputed the integral cohomology of the space $\widetilde{\mathbb{P}}^r$ using algebraic methods, in such a way that these extend to the calculation of the étale case. This is done in Chapter I for integral cohomology and in Chapter II for étale cohomology. In these two chapters we tried to be as "self contained" as possible, having in mind beginners in the subject, like young researchers from Turkey (Tuba, Ali, Handan, Yakup, ... met in Ankara, Ağustos 1995).

In Chapter III we explain by a fundamental particular case, how the results announced above for $H^*(\widetilde{\mathbb{P}}(E))$ are obtained. No proofs are given. The general case (with proofs) will be treated in [A1].

The study of $H^*(\widetilde{\mathbb{P}}(E))$ was submitted to me by Jean-Pierre JOUANOLOU. Computations in Chapters I and II were achieved under his advisement. I would like to thank him warmly here, for his encouragement and for his help.

Contents

I. Integral Cohomology of Weighted Projective Spaces

Before computing integral cohomology $H^*(-, \mathbb{Z})$ of weighted projective spaces, we first recall definitions and some properties.

Through all this chapter (I), the ground field is the complex numbers $\mathbb{C}$, with multiplicative group $\mathbb{C}^*$.

Let n be an integer ≥ 0 and fix $q_0, \cdots, q_n$, $n+1$ integers ≥ 1. The group $\mathbb{C}^*$ acts on the space $(\mathbb{C}^{n+1})^* = \mathbb{C}^{n+1} \backslash (0)$ in the following way:

$$\lambda \cdot x = (\lambda^{q_0} x_0, \cdots, \lambda^{q_n} x_n) \qquad (*)$$

where $\lambda \in \mathbb{C}^*$, $x = (x_0, \cdots, x_n) \in (\mathbb{C}^{n+1})^*$.

Notice that this action is free if, and only if, $q_0 = \cdots = q_n = 1$. We denote its quotient topological space by

$$\widetilde{\mathbb{P}}^n = \mathbb{P}_{\mathbb{C}}(q_0, \cdots, q_n)\ .$$

It is called *weighted* (or twisted) *projective space*, of type $(q_0, \cdots, q_n)$. If $q_0 = \cdots = q_n = 1$, then $\widetilde{\mathbb{P}}^n = \mathbb{P}^n$ is the usual complex projective space. For any integer $d \geq 1$, we have a natural homeomorphism $\mathbb{P}(dq_0, \cdots, dq_n) = \mathbb{P}(q_0, \cdots, q_n)$ (given by $\lambda \mapsto \lambda^d$ in $\mathbb{C}^*$).

1 First Properties of the Space $\widetilde{\mathbb{P}}^n$

(a) Local structure of $\widetilde{\mathbb{P}}^n$

For $i \in \{0, \cdots, n\}$, put

$$U_i = \{x = (x_0, \cdots, x_n) \in \widetilde{\mathbb{P}}^n \,|\, x_i \neq 0\},\ Y_i = \widetilde{\mathbb{P}}^n \backslash U_i\ .$$

Then Y_i is nothing else but $\widetilde{\mathbb{P}}^{n-1} = \mathbb{P}(q_0, \cdots, \hat{q}_i, \cdots, q_n)$ considered as a closed subset of $\widetilde{\mathbb{P}}^n$ (defined by the equation $x_i = 0$)[1]. Let us now identify the open subset $U_i \subset \widetilde{\mathbb{P}}^n$.

For an integer $q \geq 1$, consider $\mu_q \subset \mathbb{C}^*$ the subgroup of q-th roots of unity. So μ_q acts on the space $\mathbb{C}^{n+1}$ by the formula (*) above. Denote the topological quotient-space by

$$\mathbb{C}^{n+1}/\mu_q = (\mathbb{C}^{n+1}/\mu_q)\,(q_0, \cdots, q_n)\ .$$

There is a homeomorphism $U_i \simeq (\mathbb{C}^n/\mu_{q_i})(q_0, \cdots, \hat{q}_i, \cdots, q_n)$ (sending $(x_0, \cdots, \hat{x}_i, \cdots, x_n) \in \mathbb{C}^n/\mu_{q_i}$ to $(x_0, \cdots, 1, \cdots, x_n) \in U_i$, in the obvious way!).

This shows that the space $\widetilde{\mathbb{P}}^n$ is locally of the form $\mathbb{C}^n/\mu_q$ since the U_i's cover $\widetilde{\mathbb{P}}^n$. In particular, $\widetilde{\mathbb{P}}^n$ is an orbifold.

[1] $x = (x_0, \cdots, x_n)$ denotes a point in $\mathbb{C}^{n+1}$ or the class it represents in $\widetilde{\mathbb{P}}^n$. The symbol $\hat{a}$ means that the letter a is omitted.

(b) Maps between $\widetilde{\mathbb{P}}^n$ and $\mathbb{P}^n$

There are two maps, $\varphi : \mathbb{P}^n \longrightarrow \widetilde{\mathbb{P}}^n$ and $\psi : \widetilde{\mathbb{P}}^n \longrightarrow \mathbb{P}^n$, where $\widetilde{\mathbb{P}}^n = \mathbb{P}(q_0, \cdots, q_n)$ and $\mathbb{P}^n = \mathbb{P}(1, \cdots, 1)$ (the usual projective space over $\mathbb{C}$). The first one is defined by $\varphi(x_0, \cdots, x_n) = (x_0^{q_0}, \cdots, x_n^{q_n})$. For the second, put $m = \operatorname{lcm}\{q_0, \cdots, q_n\}$ and $t_i = m/q_i$. Then ψ is defined by $\psi(x_0, \cdots, x_n) = (x_0^{t_0}, \cdots, x_n^{t_n})$.

Let us make precise the fibers of φ. Consider the group $G = (\mu_{q_0} \times \cdots \times \mu_{q_n})/\mu_d$ where $d = \gcd\{q_0, \cdots, q_n\}$, and μ_d is a subgroup of $\mu_{q_0} \times \cdots \times \mu_{q_n}$ via the diagonal. There is an operation of G on $\mathbb{P}^n$ given by $(\lambda_0, \cdots, \lambda_n) \cdot (x_0, \cdots, x_n) = (\lambda_0 x_0, \cdots, \lambda_n x_n)$, with $\lambda_i \in \mu_{q_i}$, $x = (x_0, \cdots, x_n) \in \mathbb{P}^n$. This operation is faithful, i.e. no non-trivial element of G fixes all points of $\mathbb{P}^n$.

The fibers of φ are the G-orbits, and φ induces a homeomorphism $\mathbb{P}^n/G \xrightarrow{\sim} \widetilde{\mathbb{P}}^n$. Note that G has order $|G| = (\prod_0^n q_i)/d$. We will see later that φ has a degree equal to $|G|$.

The map ψ allows to define a "canonical" line bundle $\widetilde{L}$ over $\widetilde{\mathbb{P}}^n$. Let L be the canonical line bundle over the projective space $\mathbb{P}^n$. Then put $\widetilde{L} := \psi^*(L)$, the pull-back of L by ψ. Recall that $\psi^*(L)$ is defined by

$$\psi^* L = \widetilde{\mathbb{P}}^n \times_{\mathbb{P}^n} L = \{(x, u) \in \widetilde{\mathbb{P}}^n \times L \mid \psi(x) = \pi(u)\},$$

where π is the projection of L on $\mathbb{P}^n$.

(c) Another way to describe $\widetilde{L}$ is the following. Let q be an integer ≥ 1. Consider $\widetilde{\mathbb{P}}^{n+1} = \mathbb{P}(q_0, \cdots, q_{n+1})$, the (complex) weighted projective space of type $(q_0, \cdots, q_{n+1})$, with $q_{n+1} = q$. Then the space $E = \widetilde{\mathbb{P}}^{n+1} \backslash (0, \cdots, 0, 1)$ projects on $\widetilde{\mathbb{P}}^n = \mathbb{P}(q_0, \cdots, q_n)$:

$$\rho : E \longrightarrow \widetilde{\mathbb{P}}^n, \ (x_0, \cdots, x_{n+1}) \mapsto (x_0, \cdots, x_n).$$

If each q_i $(0 \leq i \leq n)$ divides q, then E is a (complex) line bundle over $\widetilde{\mathbb{P}}^n$. This is not difficult to see. The vector space structure on the fibers $\rho^{-1}(x) = E_x$ is the obvious one. To explain the local triviality, one uses the standard open subsets $U_i \subset \widetilde{\mathbb{P}}^n$ (see (a)). Let V_j be the corresponding open subsets in $\widetilde{\mathbb{P}}^{n+1}$. We have a commutative diagram $(0 \leq i \leq n)$:

$$\begin{array}{ccc} \rho^{-1}(U_i) = V_i \simeq \dfrac{\mathbb{C}^{n+1}}{\mu_{q_i}}(q_0, \cdots, \hat{q}_i, \cdots, q_{n+1}) & = & \left(\dfrac{\mathbb{C}^n}{\mu_{q_i}}(q_0, \cdots, \hat{q}_i, \cdots, q_n)\right) \times \mathbb{C} \\ \big\downarrow \rho & \swarrow & \text{1st proj.} \\ U_i \simeq \dfrac{\mathbb{C}^n}{\mu_{q_i}}(q_0, \cdots, \hat{q}_i, \cdots, q_n) & & \end{array}$$

This shows that there exists an isomorphism $E|U_i \simeq U_i \times \mathbb{C}$ over U_i (linear on the fibers).

In case $q = m = \mathrm{lcm}\{q_0, \cdots, q_n\}$ the vector bundle $\widetilde{L} = \psi^* L$ is isomorphic to E (over $\mathbb{P}^n$).

Proof. Write $L = \mathbb{P}^{n+1} \backslash (0, \cdots, 0, 1)$. Then the map

$$E \to \psi^* L = \widetilde{\mathbb{P}}^n \times_{\mathbb{P}^n} L : (x_0, \cdots, x_{n+1}) \mapsto ((x_0, \cdots, x_n), (x_0^{t_0}, \cdots, x_n^{t_n}, x_{n+1}))$$

(cf. (b) for notation) is an isomorphism of vector bundles (over $\mathbb{P}^n$).

More generally, if q is a multiple of m, and $s = q/m$, then one has $\psi^*(L^{\otimes s}) \simeq E$, since $L^{\otimes s} = \mathbb{P}^{n+1}(1, \cdots, 1, s) \backslash (0, \cdots, 0, 1)$.

(d) Consider the real sphere

$$S^{2n+1} = \{(x_0, \cdots, x_n) \in \mathbb{C}^{n+1} \mid \sum_0^n |x_i|^2 = 1\} \subset (\mathbb{C}^{n+1})^* .$$

The operation (*) of $\mathbb{C}^*$ on $(\mathbb{C}^{n+1})^*$ induces, by restriction, an operation of S^1 on S^{2n+1}. Denote the corresponding quotient-space

$$S^{2n+1}/S^1 = (S^{2n+1}/S^1)(q_0, \cdots, q_n) .$$

The natural map $S^{2n+1}/S^1 \longrightarrow \widetilde{\mathbb{P}}^n$ is a homeomorphism. In particular a weighted projective space is compact Hausdorff.

Proof. The mentioned map is induced by the closed inclusion $S^{2n+1} \subset (\mathbb{C}^{n+1})^*$. So it is continuous and closed. To see injectivity, let $x, y \in S^{2n+1}$, $\lambda \in \mathbb{C}^*$ such that: $y_i = \lambda^{q_i} x_i$ $(0 \leq i \leq n)$. Since

$\sum |x_i|^2 = 1$ and $\sum |\lambda|^{2q_i}|x_i|^2 = 1$, one has $|\lambda| = 1$. For surjectivity, given $(y_0, \cdots, y_n)$ in $(\mathbb{C}^{n+1})^*$, it is enough to find $\lambda \in \mathbb{C}^*$ such that $\sum |\lambda|^{2q_i}|y_i|^2 = 1$ (since then $(x_0, \cdots, x_n) \in S^{2n+1}$ maps to $(y_0, \cdots, y_n) \in \widetilde{\mathbb{P}}^n$, with $x_i = \lambda^{q_i} y_i$). But any equation of the form

$$a_0 t^{q_0} + \cdots + a_n t^{q_n} = 1 \quad (\text{all } a_i \geq 0, \text{ some } a_j \neq 0, \ t \in \mathbb{R})$$

has a (unique) solution $t_0 \in \mathbb{R}, t_0 > 0$. In case $a_i = |y_i|^2$, take $\lambda = \sqrt{t_0}$.

Now $\widetilde{\mathbb{P}}^n \simeq S^{2n+1}/S^1$ is Hausdorff since S^1 is a compact Hausdorff group acting on a Hausdorff space!

2 Further Properties of the Space $\widetilde{\mathbb{P}}^n$

First we establish a special cellular decomposition of $\widetilde{\mathbb{P}}^n$, where the cells have the form B^m/μ_q (B^m: real ball, μ_q: q-th roots of unity).

(a) A cellular decomposition of $\widetilde{\mathbb{P}}^n$

Here we identify $\widetilde{\mathbb{P}}^n = \mathbb{P}(q_0, \cdots, q_n)$ with S^{2n+1}/S^1 (after 1.(d)). Consider the real ball

$$B^{2n} = \{(x_0, \cdots, x_{n-1}) \in \mathbb{C}^n | \sum_0^{n-1} |x_i|^2 \leq 1\} .$$

The action $\mathbb{C}^* \times (\mathbb{C}^n)^* \longrightarrow (\mathbb{C}^n)^*$ which defines $\widetilde{\mathbb{P}}^{n-1} = \mathbb{P}(q_0, \cdots, q_{n-1})$ induces an action of the q-th roots of unity μ_q on B^{2n}, where $q = q_n$ (see (*) at the beginning). The boundary $S^{2n-1} \subset B^{2n}$ is stable under this action. So we obtain a canonical closed inclusion

$$\alpha : S^{2n-1}/\mu_q \longrightarrow B^{2n}/\mu_q ,$$

and a map $\rho : B^{2n}/\mu_q \longrightarrow S^{2n+1}/S^1$, well-defined by:

$$\rho(x_0, \cdots, x_{n-1}) = (x_0, \cdots, x_n) \text{ with } x_n = \left(1 - \sum_0^{n-1} |x_i|^2\right)^{\frac{1}{2}}$$

Let $\beta : S^{2n-1}/S^1 \longrightarrow S^{2n+1}/S^1$ be the canonical closed inclusion $\widetilde{\mathbb{P}}^{n-1} \subset \widetilde{\mathbb{P}}^n$ (defined by $x_n = 0$).

We can now state the main result in this paragraph.

Proposition *There is a commutative diagram*

$$\begin{array}{ccc} S^{2n-1}/\mu_q & \stackrel{\alpha}{\hookrightarrow} & B^{2n}/\mu_q \\ can. \downarrow \gamma & & \downarrow \rho \\ S^{2n-1}/S^1 & \underset{\beta}{\hookrightarrow} & S^{2n+1}/S^1 \end{array}$$

where ρ maps homeomorphically the open subset, complementary to $Im(\alpha)$ on the open subset, complementary to $Im(\beta)$.

This means that $\widetilde{\mathbb{P}}^n$ is obtained from $\widetilde{\mathbb{P}}^{n-1}$ by adjoining a "cell" of the form B^{2n}/μ_q. Such a structure could be called VCW-complex or orbi-CW complex (think of V-manifold, orbifold).

<u>**Proof.**</u> Commutativity of the diagram is clear. Let us see that the map ρ induces a bijection

$$\rho' : (B^{2n} \backslash S^{2n-1})/\mu_q \xrightarrow{\sim} (S^{2n+1} \backslash S^{2n-1})/S^1 \, .$$

(i) ρ' <u>is onto</u>: let $x = (x_0, .., x_n) \in S^{2n+1}$ with $x_n \neq 0$. Choose λ such that $\lambda^q = x_n/|x_n|$, and put $y_i = \lambda^{-q_i} x_i$ $(0 \leq i \leq n-1)$. Then $y = (y_0, \cdots, y_{n-1}) \in B^{2n}$, $y \notin S^{2n-1}$ and $\rho'(y) = x$.
(ii) ρ' <u>is one-one</u>: assume $\rho'(x) = \rho'(y)$. So there exists $\lambda \in S^1$ such that $y_i = \lambda^{q_i} x_i$ $(0 \leq i \leq n-1)$ and

$$\left(1 - \sum_0^{n-1} |y_i|^2\right)^{\frac{1}{2}} = \lambda^q \left(1 - \sum_0^{n-1} |x_i|^2\right)^{\frac{1}{2}} \neq 0 \, .$$

Therefore $|y_i| = |x_i|$, and so $\lambda^q = 1$. Hence $x = y$ in $(B^{2n} \backslash S^{2n-1})/\mu_q$.

Corollary *The diagram*

$$S^{2n-1}/\mu_q \xrightarrow[can.]{\gamma} S^{2n-1}/S^1 \stackrel{\beta}{\hookrightarrow} S^{2n+1}/S^1$$

is a cofibration, that is S^{2n+1}/S^1 is the mapping cone $C(\gamma)$ of γ and β is the canonical injection of S^{2n-1}/S^1 into $C(\gamma)$.

It is enough to see that B^{2n}/μ_q is the cone of $S^{2n-1}/\mu_q =: X$, with canonical inclusion α. This is given by the homeomorphism

$$CX = (X\times I)/X\times\{0\} \to B^{2n}/\mu_q : ((x_0,\cdots,x_{n-1}),t) \mapsto (tx_0,\cdots,tx_{n-1})$$

where $I=[0,1]$.

REMARK. From the preceding we deduce (1.(a))

$$\mathbb{C}^n/\mu_q = (\mathbb{C}/\mu_q)\,(q_0,..,q_{n-1}) = \widetilde{\mathbb{P}}^n \backslash \widetilde{\mathbb{P}}^{n-1} = CX\backslash X$$

where $X=S^{2n-1}/\mu_q$, $CX=$ cone of $X, q=q_n$. The point $0\in\mathbb{C}^n/\mu_q$ corresponds to the vertex of CX.

(b) Degree of the map φ

We show that the map $\varphi : \mathbb{P}^n \longrightarrow \widetilde{\mathbb{P}}^n$ has a degree, and that the degree is equal to $(\prod_0^n q_i)/d$ (cf. 1.(b)).

Because of what we saw in 1.(b), our situation is a particular case of the following.

Theorem *Let X be an orientable connected manifold. Assume a finite group G acts faithfully on X, and that G preserves the orientation. Then*
(i) $H_c^r(X/G,\ \mathbb{Z})=\mathbb{Z}$ ($r=\dim X, H_c^{\cdot}=$ cohomology with compact supports).
(ii) the canonical projection $\pi : X \longrightarrow X/G$ has a degree equal to $|G|$.

This result must be "well-known", but we do not know a reference for a proof.

<u>**Proof.**</u> (i) We may assume $G\neq\{1\}$. Let $g\in G$, $g\neq 1$. Then $X^g=\{x\in X \,|\, gx=x\}$ is a closed submanifold of X, with dimension $\leq r-1$. The open subset $U=X\backslash\cup_{h\neq 1}X^h$ is non-empty, G-invariant, and the action of G on U is free. Assume for a while that we have proved $\dim(X^g)\leq r-2$. Put $Y=\cup_{h\neq 1}X^h$. The Mayer-Vietoris exact sequence ([B] II.13.(1)) shows that $H_c^i(Y\mathbb{Z})=0$ for $i\geq r-1$.

Hence we obtain the following commutative diagram (cohomology groups have coefficients in $\mathbb{Z}$).

$$\begin{array}{ccc} H^r_c(U) & \xrightarrow{\sim} & H^r_c(X) = \mathbb{Z} \\ {\scriptstyle (\pi_U)^*}\uparrow & & \uparrow{\scriptstyle \pi^*} \\ H^r_c(U/G) & \xrightarrow{\sim} & H^r_c(X/G) \to 0 \\ \| & & \\ \mathbb{Z} & & \end{array} \qquad (U \text{ is connected; } \pi, \pi_u \text{ are proper})$$

So $H^r_c(X/G)$ has the form $\mathbb{Z}/a\mathbb{Z}$ $(a \geq 0)$. But $\pi^* \otimes \mathbb{Q}$ is an isomorphism ([B] II (19.1)), whence $a = 0$. This proves (i).

It remains to show dim $(X^g) \leq r-2$. We can assume $X^g \neq \emptyset$. Consider a point $x \in X^g$. The subgroup (g), spanned by g, acts on the tangent space $T_x(X)$ (by automorphisms), and one has $T_x(X^g) = T_x(X)^g$. Let us assume that dim $(X^g) = r-1$. It follows that $T_x(X)/T_x(X)^g = \mathbb{R}$ and that (g) acts on $\mathbb{R}$ (by homotheties), preserving orientation. Now g has finite order, so it acts on $\mathbb{R}$ as a homothety the coefficient of which is 1. Hence $T_x(X)^g = T_x(X)$ (The action of (g) on $T_x(X)$ is semi-simple, by Maschke theorem). This gives a contradiction!

(ii) The commutative diagram above shows it is enough to see that $\rho = \pi_U : U \longrightarrow U/G$ has a degree equal to $|G|$. Since G acts freely on U, there exists a non-empty open subset $V \subset U$ such that $V = \cup V_g$ $(g \in G)$, where the V_g's are connected disjoint open subsets with the property:

$$\rho_g = \rho_{V_g} : V_g \longrightarrow V/G \text{ is a homeomorphism for all } g \in G.$$

One obtains a commutative diagram (coefficients are in $\mathbb{Z}$) of cohomology groups

$$\begin{array}{ccccc}
H_c^r(V) & & \longrightarrow & & \\
 & \diagdown\!\!\diagdown\, (=) & & & \\
\Big\uparrow \rho_V^* & & \bigoplus_g H_c^r(V_g) & \longrightarrow & H_c^r(U) \\
 & & \Big\uparrow \oplus\rho_g^* & & \Big\uparrow \rho^* \\
 & & H_c^r(V/G) & \xrightarrow{\sim} & H_c^r(U/G) ,
\end{array}$$

the square of which is

$$\begin{array}{ccc}
\bigoplus_g \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
\Big\uparrow \Delta & & \Big\uparrow \mu \\
\mathbb{Z} & \xrightarrow[=]{} & \mathbb{Z}
\end{array} \quad \text{with } \Delta(z) = (z, \cdots, z),\ \sigma(z_1, \cdots, z_s) = z_1 + \cdots + z_s ,$$

where $s = |G|$. This implies that μ is multiplication by s, as desired.

(c) We shall need the following topological property of $\widetilde{\mathbb{P}}^n$.

In the space $\widetilde{\mathbb{P}}^n$ every point has a fundamental system of contractible open neighborhoods.

Since $\widetilde{\mathbb{P}}^n$ is homeomorphic to $\mathbb{P}^n/G$ (see 1.(b)), our assertion is a corollary of the more general following fact.

Proposition *Let X be a manifold on which a finite group G acts. Then every point in the quotient-space X/G has a fundamental system of contractible open neighborhoods.*

Proof. Take $\bar{x} \in X/G$ and an open subset $\bar{U} \subset X/G$ with $\bar{x} \in \bar{U}$. Let U be the inverse image of $\bar{U}$ in X and x a point in the fiber over $\bar{x}$. Then there exists ([HM] 1.6. SATZ page 7) a neighborhood V of x, $V \subset U$, and a coordinate system

$$f = (f_1, \cdots, f_r) : V \longrightarrow \mathbb{R}^r$$

such that $f(x) = 0$, $f(V)$ is an open ball with center 0 and the following are satisfied.

(i) the stabilizing subgroup G_x acts linearly on $\mathbb{R}^r$, V is G_x-invariant and f is G_x-equivariant;
(ii) there exists a G_x-invariant linear subspace L in $\mathbb{R}^r$ so that $T = f^{-1}(f(V) \cap L)$ is a "slice" of U at the point x,
i.e. (1) $x \in T$ and T is stable under the action of G_x,
(2) for every $g \in G$, if $(gT) \cap T \neq \emptyset$ then $g \in G_x$,
(3) for every local section $s : E \longrightarrow G$ of the canonical projection $G \longrightarrow G/G_x$, where E contains the identity element, the map

$$E \times T \longrightarrow U : (e, t) \mapsto s(e)t$$

is a diffeomorphism on an open subset of U.

Now $(f(V) \cap L)/G_x$ is contractible, so T/G_x is also contractible. On the other hand, the projection $\pi : U \longrightarrow \bar{U} = U/G$ induces a map $T/G_x \longrightarrow U/G$ which is a homeomorphism on $\pi(T)$ ([HM] 1.6. FOLGERUNG page 8). But $\pi(T)$ is an open subset in $\bar{U}$ containing $\bar{x}$ (in (ii)(3) above, take $E = \{\bar{1}\}$ and $s(\bar{1}) = 1$), and $\pi(T)$ is contractible.

(d) Consider the open subset $\widetilde{\mathbb{P}}^n = \mathbb{P}(q_0, \cdots, q_n)$

$$\mathbb{C}^n/\mu_q = (\mathbb{C}^n/\mu_q)\,(q_0, \cdots, q_{n-1})$$

where $q = q_n$ (see 1.(b)). We want to show that the natural map

$$H_0^*(\mathbb{C}^n/\mu_q,\ \mathbb{Z}) \longrightarrow H_c^*(\mathbb{C}^n/\mu_q,\ \mathbb{Z})$$

is an isomorphism (H_0^*= local cohomology at the point $0 = (0, \cdots, 0) \in \mathbb{C}^n/\mu_q$, H_c^*=cohomology with compact supports).

The remark in (a) above shows that this result is a particular case of the following one.

Proposition *Take a compact topological space X and put*

$$Y = CX \backslash X, \text{ where } CX = \text{ cone of } X\ .$$

Call 0 the vertex of CX. Let $\mathfrak{F}$ be a constant abelian sheaf over CX. Then the natural homomorphism

$$H_0^*(U,\ \mathfrak{F}|U) \longrightarrow H_c^*(U,\ \mathfrak{F}|U)$$

is bijective.

Proof. Denote by $i : U \longrightarrow CX$ the natural inclusion of U in CX. For any abelian sheaf $\mathcal{A}$ over CX, one has a morphism between exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Gamma_0(CX,\mathcal{A}) & \longrightarrow & \Gamma(CX,\mathcal{A}) & \longrightarrow & \Gamma(CX\backslash 0,\mathcal{A}) \\
 & & \downarrow & & \downarrow \| & & \downarrow \\
0 & \longrightarrow & \Gamma(CX, i_!(\mathcal{A}|\mathcal{U})) & \longrightarrow & \Gamma(CX,\mathcal{A}) & \longrightarrow & \Gamma(X,\mathcal{A}|\mathcal{X})
\end{array}$$

where $i_!$ means extension by zero. Apply this morphism to an injective resolution of $\mathfrak{F}$ and take cohomology. One obtains the following exact sequence morphism:

$$\begin{array}{ccccccccc}
\cdots \to & H_0^j(CX,\mathfrak{F}) & \to & H^j(CX,\mathfrak{F}) & \to & H^j(CX\backslash 0,\mathfrak{F}) & \to & H_0^{j+1}(CX,\mathfrak{F}) & \to \cdots \\
 & \downarrow & & \downarrow \| & & \downarrow \sim & & \downarrow & \\
\cdots \to & H_c^j(U,\mathfrak{F}|U) & \to & H^j(CX,\mathfrak{F}) & \to & H^j(X,\mathfrak{F}|X) & \to & H_0^{j+1}(U,\mathfrak{F}|U) & \to \cdots
\end{array}$$

(the inclusion $X \subset CX\backslash 0$ is a homotopy equivalence). By the five lemma it follows that we have isomorphisms

$$H_0^j(CX,\mathfrak{F}) \xrightarrow{\sim} H_c^j(U,\mathfrak{F}|U) \quad (j \geq 0)\ .$$

But the homomorphisms $H_0^j(CX,\mathfrak{F}) \longrightarrow H_0^j(U,\mathfrak{F}|U)$ are bijective (U is open in CX and $0 \in U$), and there are commutative diagrams

$$\begin{array}{ccc}
H_0^j(U,\mathfrak{F}|U) & \longrightarrow & H_c^j(U,\mathfrak{F}|U) \\
\nwarrow \sim & & \sim \nearrow \\
 & H_0^j(CX,\mathfrak{F}) &
\end{array}$$

which ends the proof.

3 Grothendieck Theorem

Here we state some notation, and recall a theorem by GROTHENDIECK we shall use later.

Let X be a topological space on which a group G acts. Denote by $\pi : X \longrightarrow Y = X/G$ the canonical inclusion. Let $\mathfrak{F}$ be an abelian G-sheaf over X. Put

$$\Gamma_X^G(\mathfrak{F}) = \Gamma(X, \mathfrak{F})^G \quad \text{(invariant sections)}$$

and write $\pi_*^G(\mathfrak{F})$ for the sheaf over Y

$$V \mapsto \Gamma(V, \pi_*\mathfrak{F})^G \ , \ V \text{ open subset in } Y \ ,$$

(G acts identically on Y, $\pi_*\mathfrak{F}$ is a G-sheaf). So one has two left exact functors

$$\mathfrak{F} \mapsto \Gamma_X^G(\mathfrak{F}) \ , \ \mathfrak{F} \mapsto (\pi_*\mathfrak{F})^G \ ,$$

and can consider their derived functors

$$H^i(X; G, \mathfrak{F}) := R^i\Gamma_X^G(\mathfrak{F}) \ , \ \mathfrak{H}^i(G, \mathfrak{F}) := R^i\Pi_*^G(\mathfrak{F}) \quad (i \geq 0) \ .$$

Alexander GROTHENDIECK established the following results [G2].
(a) Theorem ([G2]5.2.1) *There are two spectral sequences*

$$\begin{aligned} {}'E_2^{i,j} &= H^i(Y, \mathfrak{H}^j(G, \mathfrak{F})) \Longrightarrow H^{i+j}(X; G, \mathfrak{F}) \ , \\ {}''E_2^{i,j} &= H^i(G, H^j(X, \mathfrak{F})) \Longrightarrow H^{i+j}(X; G, \mathfrak{F}) \ . \end{aligned}$$

Let us call (D) the following condition:

> for every $x \in X$, the stabilizing subgroup G_x of x is finite, and there exists a neighborhood V_x of x such that $(gV_x) \cap V_x = \emptyset$ for all $g \in G \backslash G_x$.

For instance (D) is verified if X is Hausdorff and G is finite.

(b) Theorem ([G2] 5.3.1) *If the condition (D) is satisfied, then there are natural isomorphisms*

$$\mathfrak{H}^i(G, \mathfrak{F})_y = H^i(G_x, \mathfrak{F}_x)$$

for all $y \in Y$ and all $x \in X$ with $\pi(x) = y$.

(c) Corollary ([G2] Corollaire 1 to 5.3.1) *Assume (D) is true. If, for every $x \in X$, multiplication by $|G_x|$ is a group automorphism of $\mathfrak{F}_x$, then there is a spectral sequence*

$$''E_2^{i,j} = H^i(G, H^j(X, \mathfrak{F})) \Longrightarrow H^{i+j}(Y, \pi_*^G \mathfrak{F})$$

(by Theorem (b), $\mathfrak{H}^i(G, \mathfrak{F}) = 0$ for $i > 0$; apply then Theorem (a)).

The following theorem is a consequence of Corollary (c) ([G2] Corollaire to 5.2.3).

(d) GROTHENDIECK Theorem: *Assume G finite and X Hausdorff. If multiplication by the order of G is an automorphism of $\mathfrak{F}$, then*

$$H^j(Y, \pi_*^G \mathfrak{F}) = H^j(X, \mathfrak{F})^G \quad (j \geq 0)\ .$$

4 The Groups $H^i(\widetilde{\mathbb{P}}^n, \mathbb{Z})$

Consider an integer $a \geq 0$, and put $A = \mathbb{Z}/a\mathbb{Z}$. If X is a topological space, we write A_X, or simply A, for the constant abelian sheaf defined by A over X. For instance, cohomology groups of X with values in A_X are denoted by $H^i(X, A)$ $(i \geq 0)$.

Recall that $\widetilde{\mathbb{P}}^n$ is the weighted projective space $\mathbb{P}(q_0, \cdots, q_n)$ over the complex numbers $\mathbb{C}$, defined in the beginning. The main result in this paragraph is the following.

(a)Theorem *We have*

$$H^i(\widetilde{\mathbb{P}}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 2r,\ 0 \leq r \leq n\ , \\ 0 & \text{if } i \text{ is odd or } i > 2n. \end{cases}$$

To prove this problem we first compute the groups $H^i(\widetilde{\mathbb{P}}^n, \mathbb{Z}/a\mathbb{Z})$ where $a > 0$, and then deduce the groups $H^i(\widetilde{\mathbb{P}}^n, \mathbb{Z})$ by use of the universal-coefficient formula. Denote $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$.

Lemma 1 For any integer $a > 0$, one has

$$H^i(\widetilde{\mathbb{P}}^n, \mathbb{Z}_a) = \begin{cases} \mathbb{Z}_a & \text{if } i = 2r,\ 0 \leq r \leq n\ , \\ 0 & \text{if } i \neq 2r,\ 0 \leq r \leq n. \end{cases}$$

<u>**Proof.**</u> We may assume $a = p^\alpha$ with p prime number $\neq 1$ and $\alpha > 0$ (if not, decompose $\mathbb{Z}_a$!). Because $\mathbb{P}(dq_0, \cdots, dq_n) = \mathbb{P}(q_0, \cdots, q_n)$ we may also assume gcd $\{q_0, \cdots, q_n\} = 1$. Now, let us proceed by induction on n. The case $n = 0$ is clear since $\widetilde{\mathbb{P}}^0$ is a point. So let $n \geq 1$. There exists j, $0 \leq j \leq n$, such that p does not divide q_j. Consider then

$$\widetilde{\mathbb{P}}^{n-1} = \mathbb{P}(q_0, .., \hat{q}_j, \cdots, q_n), \mathbb{C}^n/\mu_q = (\mathbb{C}^n/\mu_{q_j})(q_0, \cdots, \hat{q}_j, \cdots, q_n)(\text{cf. 1.(a)}).$$

Put $q = q_j$. Since $\mathbb{C}^n/\mu_q$ is the complement of the closed subset $\widetilde{\mathbb{P}}^{n-1} \subset \widetilde{\mathbb{P}}^n$, there is a cohomology exact sequence, with compact supports, and coefficients in $\mathbb{Z}_{p^\alpha}$,

$$\cdots \to H^i_c(\mathbb{C}^n/\mu_q) \to H^i(\widetilde{\mathbb{P}}^n) \to H^i(\widetilde{\mathbb{P}}^{n-1}) \to H^{i+1}_c(\mathbb{C}^n/\mu_q) \to \cdots$$

Hence it is enough to show the following, and to use induction.

Lemma 2 We have

$$H^i_c(\mathbb{C}^n/\mu_q, \mathbb{Z}_{p^\alpha}) = \begin{cases} \mathbb{Z}_{p^\alpha} & \text{if } i = 2n, \\ 0 & \text{if } i \neq 2n. \end{cases}$$

where $(p, q) = 1$.

<u>**Proof.**</u> Lemma 2 is well-known in case $q = 1$. So it suffices to show:

$$H^i_c(\mathbb{C}^n/\mu_q, \mathbb{Z}_{p^\alpha}) = [H^i_c(\mathbb{C}^n, \mathbb{Z}_{p^\alpha})]^{\mu_q} \quad (i \geq 0) \tag{1}$$

(since μ_q acts identically on $H^i_c(\mathbb{C}^n, \mathbb{Z}_{p^\alpha})$). Consider the open inclusion in the usual projective space

$$\mathbb{C}^n \hookrightarrow \mathbb{P}^n : (x_0, \cdots, \hat{x}_j, \cdots, x_n) \mapsto (x_0, \cdots, 1, \cdots, x_n) \text{ (1 is at the } j\text{-th place)},$$

and the action of μ_q on $\mathbb{P}^n$ is defined by

$$\lambda \cdot (x_0, \cdots, x_n) = (\lambda^{q_0} x_0, \cdots, \lambda^{q_n} x_n)\ .$$

The inclusion above is then μ_q-equivariant and there is a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^n & \overset{}{\underset{\text{open}}{\hookrightarrow}} & \mathbb{P}^n \\ {\scriptstyle \text{can.}}\downarrow & & \downarrow{\scriptstyle \text{can.}} \\ \mathbb{C}^n/\mu_q & \underset{\text{open}}{\hookrightarrow} & \mathbb{P}^n/\mu_q \ . \end{array}$$

Therefore (1) is a particular case of the following proposition.

Proposition *Let X be a compact Hausdorff topological space on which a finite group G acts. Let U be a G-stable open subset of X with canonical projection $\pi : U \longrightarrow U/G$. Suppose $\mathfrak{F}$ is an abelian G-sheaf over U such that the multiplication by the order of G is an automorphism of $\mathfrak{F}$. Then*

$$H_c^i(U/G, \pi_*^G \mathfrak{F}) = [H_c^i(U, \mathfrak{F})]^G \quad (i \geq 0)$$

(for notation see 3.)

Proof. There is a commutative diagram, with obvious notation,

$$\begin{array}{ccc} U & \overset{i}{\hookrightarrow} & X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \sigma} \\ U/G & \underset{\bar{i}}{\hookrightarrow} & X/G \ . \end{array}$$

We have

$$H_c^k(U, \mathfrak{F}) = H^k(X, i_! \mathfrak{F}), \ H_c^k(U/G, \pi_*^G \mathfrak{F}) = H^k(X/G, \bar{i}_! \pi_*^G \mathfrak{F}),$$

where $i_!$, $\bar{i}_!$ are extension by 0. Multiplication by $|G|$ is an automorphism of $i_! \mathfrak{F}$ ($i_!$ is exact). Therefore, Grothendieck theorem shows that we have only to verify (3.(d))

$$\bar{i}_! \pi_*^G \mathfrak{F} = \sigma_*^G i_! \mathfrak{F} \ .$$

But one has

$$\bar{i}^* \sigma_*^G i_! \mathfrak{F} = \pi_*^G i^* i_! \mathfrak{F} = \pi_*^G \mathfrak{F} \ ,$$

and, because $\bar{\imath}_!$ is left adjoint for $\bar{\imath}^*$, one obtains a natural morphism

$$\bar{\imath}_!\pi_*^G\mathfrak{F} \longrightarrow \sigma_*^G i_!\mathfrak{F} ,$$

which is an isomorphism over the open subset U/G of X/G. To see that is an isomorphism over the whole X/G, it is enough to check

$$(\sigma_*^G i_!\mathfrak{F})_y = 0 \text{ for all } y \in X/G,\ y \notin U/G .$$

But $\sigma_*^G i_!\mathfrak{F}$ is a subsheaf of $\sigma_* i_!\mathfrak{F}$ and

$$(\sigma_* i_!\mathfrak{F})_y = H^0(\sigma^{-1}(y), (i_!\mathfrak{F})|\sigma^{-1}(y)) \quad (\sigma \text{ is proper!}) ,$$

and $(i_!\mathfrak{F})|\sigma^{-1}(y) = 0$ $(\sigma^{-1}(y) \subset X\backslash U)$.

The proof of the Proposition is finished. Hence Lemma 1 and Lemma 2 are proved.

(b) Proof of the Theorem

To simplify notation, we put $H^i(A) = H^i(\widetilde{\mathbb{P}}^n, A)$, where $A = \mathbb{Z}$ or $\mathbb{Z}_a$ $(a \geq 1)$. The exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z} \to \mathbb{Z}_a \to 0$ induces a cohomology exact sequence

$$\cdots \longrightarrow H^i(\mathbb{Z}) \xrightarrow{\cdot a} H^i(\mathbb{Z}) \longrightarrow H^i(\mathbb{Z}_a) \longrightarrow H^{i+1}(\mathbb{Z}) \xrightarrow{\cdot a} \cdots .$$

which in turn gives the following exact sequences

$$0 \longrightarrow H^i(\mathbb{Z}) \otimes \mathbb{Z}_a \longrightarrow H^i(\mathbb{Z}_a) \longrightarrow \operatorname{Tor}(H^{i+1}(\mathbb{Z}), \mathbb{Z}_a) \longrightarrow 0 .$$

By [D] 3.8., these sequences split. This proves the universal coefficient formula:

$$H^i(\mathbb{Z}_a) = (H^i(\mathbb{Z}) \otimes \mathbb{Z}_a) \oplus \operatorname{Tor}(H^{i+1}(\mathbb{Z}), \mathbb{Z}_a) . \tag{2}$$

The groups $H^i(\mathbb{Z})$ are finitely generated (2.(c) and [B] II.16.5). Fix $i \geq 0$ and write $H^i(\mathbb{Z})$ as

$$\mathbb{Z}^\beta \oplus \mathbb{Z}_{\beta_1} \oplus \cdots \oplus \mathbb{Z}_{\beta_s} \quad (\beta \geq 0, \beta_k \geq 1).$$

(i) If i is odd or $i > 2n$, then Lemma 1 and (2) imply

$$\mathbb{Z}_a \otimes H^i(\mathbb{Z}) = 0 \text{ for all } a > 0 ,$$

and so: $\beta = 0, \beta_1 = \cdots = \beta_s = 1$, i.e. $H^i(\mathbb{Z}) = 0$.
(ii) If $i = 2r$, $0 \leq r \leq n$, then Lemma 1, (2) and (i) imply

$$\mathbb{Z}_a = \mathbb{Z}_a \otimes H^i(\mathbb{Z}) \text{ for all } a > 0 .$$

Whence $\beta = 1$ (choose a prime with all the β_k's). So

$$\mathbb{Z}_a = \mathbb{Z}_a \oplus \mathbb{Z}_{(a,\beta_1)} \oplus \cdots \oplus \mathbb{Z}_{(a,\beta_s)} ,$$

which means $(a, \beta_1) = 1, \cdots, (a, \beta_s) = 1$, for all $a > 0$. This gives $\beta_1 = 1, \cdots, \beta_s = 1$, i.e. $H^i(\mathbb{Z}) = \mathbb{Z}$. The theorem is proved.

5 The Homomorphism φ^*

Our purpose here is to compute the homomorphism $\varphi^* : H^*(\widetilde{\mathbb{P}}^n, \mathbb{Z}) \longrightarrow H^*(\mathbb{P}^n, \mathbb{Z})$ induced by the map $\varphi : \mathbb{P}^n \longrightarrow \widetilde{\mathbb{P}}^n$ (see 1.(b)). We need first to define some notations.
Take $k \in \{0, \cdots, n\}$ and consider

$$I = \{i_0, \cdots, i_k\} \text{ with } 0 \leq i_0 < \cdots < i_k \leq n .$$

Then put

$$l_I = l_I(q_{i_0}, \cdots, q_{i_k}) = q_{i_0} q_{i_1} \cdots q_{i_k} / (q_{i_0}, \cdots, q_{i_k}) ,$$

where $(a_0, \cdots, a_k) = \gcd\{a_0, \cdots, a_k\}$. The following integers will play an important role in the sequel.

$$l_k = l_k(q_0, \cdots, q_n) = \operatorname{lcm}\{l_I | I \subset \{0, \cdots, n\}, |I| = k+1\} .$$

For instance $l_0 = 1, l_1 = \operatorname{lcm}\{q_0, \cdots, q_n\}$ and $l_n = q_0 \cdots q_n / \gcd\{q_0, \cdots, q_n\}$. Notice that $l_n = \deg(\varphi)$ (2.(b)). To compute the integer l_k, consider $E = \{p \text{ prime } | p \text{ divides some } q_i, \ 0 \leq i \leq n\}$. Then

$$l_k = \prod_{p \in E} p^{\alpha(p,k)} ,$$

where $\alpha(p, k) = v_p(l_k)$ is determined as follows. Put $\alpha_i = v_p(q_i)$ and order the α_i's: $\alpha_{i_0} \leq \alpha_{i_1} \leq \cdots \leq \alpha_{i_n}$ ($p \in E$ is fixed). Then

$$\alpha(p, k) = \alpha_{i_n} + \cdots + \alpha_{i_{n-k+1}} \quad (0 \leq k \leq n) .$$

Let $\xi = c_1(L^*)$ be the usual generator of $H^2(\mathbb{P}^n, \mathbb{Z})$ (L is the canonical bundle over $\mathbb{P}^n$). It is well-known that $\{1, \xi, \cdots, \xi^n\}$ is a $\mathbb{Z}$-basis of $H^*(\mathbb{P}^n, \mathbb{Z})$ [BO].

Theorem *For each k, $0 \leq k \leq n$, there exists a unique $\xi_k \in H^{2k}(\widetilde{\mathbb{P}}^n, \mathbb{Z})$ such that $\varphi^*(\xi_k) = l_k \xi^k$, and $\{1, \xi, \cdots, \xi^n\}$ is a $\mathbb{Z}$-basis of the free abelian group $H^*(\widetilde{\mathbb{P}}^n, \mathbb{Z})$(cf. 4.(a)). In other words there are commutative diagrams*

$$\begin{array}{ccc} H^{2k}(\widetilde{\mathbb{P}}^n, \mathbb{Z}) & \xrightarrow{\varphi^*} & H^{2k}(\mathbb{P}^n, \mathbb{Z}) \quad (0 \leq k \leq n) \\ \| & & \| \\ \mathbb{Z} & \xrightarrow[\cdot l_k]{} & \mathbb{Z} \end{array}$$

<u>**Proof.**</u> Let $k \in \{0, \cdots, n\}$, fixed in all the proof. For $I = \{i_0, \cdots, i_k\}$, with $0 \leq i_0 < \cdots < i_k \leq n$, we put

$$\mathbb{P}_I^k = \mathbb{P}_{\mathbb{C}}(q_{i_0}, \cdots, q_{i_k}) ,$$

and denote by u_I the closed inclusion $\mathbb{P}_I^k \subset \widetilde{\mathbb{P}}^n$ defined by $x_i = 0$ for all $i \notin I$. Let m_I be the integer defined by the diagram

$$\begin{array}{ccc} H^{2k}(\widetilde{\mathbb{P}}^n, \mathbb{Z}) & \xrightarrow{u_I^*} & H^{2k}(\widetilde{\mathbb{P}}^k, \mathbb{Z}) \\ \| & & \| \\ \mathbb{Z} & \xrightarrow[\cdot m_I]{} & \mathbb{Z} \end{array}$$

Lemma 1: *The integers, elements of the set*

$$M = \{m_I | I \subset \{0, \cdots, n\},\ |I| = k+1\} ,$$

have gcd equal to 1.

<u>**Proof of Lemma 1**</u>. We use induction on n. If $n = 0$, then $M = \{1\}$, and the lemma is clear. Assume then $n \geq 1$ and the result true in case $n-1$. If $k = n$, then $M = \{1\}$. So we can suppose $k \leq n-1$. Now, for $0 \leq i \leq n$, consider $\widetilde{\mathbb{P}}_i^{n-1} = \mathbb{P}_{\mathbb{C}}(q_0, \cdots, \hat{q}_i, \cdots, q_n)$, $u_i : \widetilde{\mathbb{P}}_i^{n-1} \subset \widetilde{\mathbb{P}}^n$ the inclusion defined by $x_i = 0$. Denote by m_i the integer defined by the diagram

$$\begin{array}{ccc} H^{2k}(\widetilde{\mathbb{P}}^n, \mathbb{Z}) & \xrightarrow{u_i^*} & H^{2k}(\widetilde{\mathbb{P}}_i^{n-1}, \mathbb{Z}) \\ \parallel & & \parallel \\ \mathbb{Z} & \xrightarrow[\cdot m_I]{} & \mathbb{Z} \end{array}$$

Take $I = \{i_0, \cdots, i_k\}$, with $0 \leq i_0 < \cdots < i_k \leq n$, and i, $0 \leq i \leq n$, such that $i \notin I$. Then the map u_I factorizes through $\widetilde{\mathbb{P}}_i^{n-1}$:

$$\begin{array}{ccc} \widetilde{\mathbb{P}}_I^k & \xrightarrow{u_I} & \widetilde{\mathbb{P}}^n \\ & u_{I,i} \nwarrow \quad \nearrow u_i & \\ & \widetilde{\mathbb{P}}_i^{n-1} & \end{array}$$

Hence $u_I^* = u_{I,i}^* \circ u_i^*$, which shows that $m_I = m_{I,i} m_i$. Therefore M is the set of integers m such that:

> there exist $I \subset \{0, \cdots, m\}$, $|I| = k+1$, and i, $0 \leq i \leq$, $i \notin I$ verifying $m = m_i m_{I,i}$.

(Recall $k \leq n-1$). But, by induction hypothesis, for each i, $0 \leq i \leq n$, the elements of the set

$$M_i = \{M_{I,i} | I \subset \{0, \cdots, n\},\ i \notin I,\ |I| = k+1\}$$

have gcd equal to 1. So we need only to show the next lemma.

Lemma 2: *One has* $gcd\{m_0, \cdots, m_n\} = 1$.

Proof of Lemma 2. Since $\mathbb{P}(dq_0, \cdots, dq_n) = \mathbb{P}(q_0, \cdots, q_n)$, we may assume $\gcd\{q_0, \cdots, q_n\} = 1$. The inclusion $u_i : \widetilde{\mathbb{P}}_i^{n-1} \subset \widetilde{\mathbb{P}}^n$ has $\mathbb{C}^n/\mu_q$ as complementary open set (1.(a)), showing that there is an exact sequence (with coefficients in $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$, $a \geq 0$):

$$\begin{array}{ccccccc} H^{2k}(\widetilde{\mathbb{P}}^n) & \xrightarrow{u_i^*} & H^{2k}(\widetilde{\mathbb{P}}_i^{n-1}) & \longrightarrow & H_c^{2k+1}(\mathbb{C}^n/\mu_q) & \longrightarrow & 0 \\ \parallel & & \parallel & & & & \\ \mathbb{Z}_a & \xrightarrow[\cdot m_i]{} & \mathbb{Z}_a & & & & \end{array}$$

Assume a is a prime number which does not divide q_i. Then, by 4.Lemma 2, $H_c^{2k+1}(\mathbb{C}^n/\mu_{q_i}, \mathbb{Z}_a) = 0$, which means that multiplication by m_i is an automorphism of $\mathbb{Z}_a$. Hence a does not divide m_i. This implies $\gcd\{m_0, \cdots, m_n\} = 1$.

To prove the theorem, take $I = \{i_0, \cdots, i_k\}$ with $0 \le i_0 < \cdots < i_k \le n$ and consider the commutative diagram

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\varphi} & \widetilde{\mathbb{P}}^n \\ \uparrow & & \uparrow u_I \\ \mathbb{P}^k & \xrightarrow[\varphi_I]{} & \widetilde{\mathbb{P}}^k_I \end{array} \quad , \text{ where } \varphi_I(x_{i_0}, \cdots, x_{i_k}) = (x_{i_0}^{q_{i_0}}, \cdots, x_{i_k}^{q_{i_k}}) \ .$$

Thus induces a cohomology commutative diagram (coefficients in $\mathbb{Z}$)

$$\begin{array}{ccc} H^{2k}(\widetilde{\mathbb{P}}^n) & \xrightarrow{\varphi^*} & H^{2k}(\mathbb{P}^n) \\ u_I^* \downarrow & & \downarrow \\ H^{2k}(\widetilde{\mathbb{P}}^k_I) & \xrightarrow[\varphi_I^*]{} & H^{2k}(\mathbb{P}^k) \end{array} \quad = \quad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{\cdot a} & \mathbb{Z} \\ \cdot m_i \downarrow & & \downarrow \| \\ \mathbb{Z} & \xrightarrow[\cdot a_I]{} & \mathbb{Z} \end{array}$$

Whence $a = m_I a_I$, for all $I \subset \{0, \cdots, n\}$ with $|I| = k + 1$. Lemma 1 implies then $\operatorname{lcm}\{a_i | I \subset \{0, \cdots, n\}, \ |I| = k + 1\}$. But $a_I = \deg\,(\varphi_I) = q_{i_0} \cdots q_{i_k} / (q_{i_0}, \cdots, q_{i_k})$ (2.(b)). The proof of the theorem is finished.

Consider the closed inclusion $\widetilde{\mathbb{P}}^{n-1} = \mathbb{P}(q_0, \cdots, q_{n-1}) \subset \widetilde{\mathbb{P}}^n$ $(n \ge 1)$ and its complementary

$$\mathbb{C}^n/\mu_q = (\mathbb{C}^n/\mu_q)(q_0, \cdots, q_{n-1}), \text{ where } q = q_n \quad \text{(cf. 1.(a))} \ .$$

For $0 \le k \le n - 1$, consider the integer (see beginning of this §5.)

$$m_k = l_k(q_0, \cdots, q_n) / l_k(q_0, \cdots, q_{n-1}) \ .$$

Corollary *The cohomology groups, with compact supports, of $\mathbb{C}^n/\mu_q$ are:*

$$H^i_c(\mathbb{C}^n/\mu_q, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 2n \ , \\ \mathbb{Z}/m_k\mathbb{Z} & \text{if } i = 2k+1, \ 1 \le k \le n-1 \ , \\ 0 & \text{if not.} \end{cases}$$

Proof. There are exact sequences (coefficients in $\mathbb{Z}$)

$$0 \to H_c^{2i}(\mathbb{C}^n/\mu_q) \to H^{2i}(\widetilde{\mathbb{P}}^n) \to H^{2i}(\widetilde{\mathbb{P}}^{n-1}) \to H_c^{2i+1}(\mathbb{C}^n/\mu_q) \to 0 .$$

On the other hand, putting $f = \varphi(q_0, \cdots, q_n) : \mathbb{P} \longrightarrow \widetilde{\mathbb{P}}^n$, $g = \varphi(q_0, \cdots, q_{n-1}) : \mathbb{P}^{n-1} \longrightarrow \widetilde{\mathbb{P}}^{n-1}$ (1.(b)), we have commutative squares

$$\begin{array}{ccc} H^{2i}(\widetilde{\mathbb{P}}^n) & \longrightarrow & H^{2i}(\widetilde{\mathbb{P}}^{n-1}) \\ f^* \downarrow & & \downarrow g^* \\ H^{2i}(\mathbb{P}^n) & \longrightarrow & H^{2i}(\mathbb{P}^{n-1}) \end{array}$$

Then apply the theorem above.

6 The Ring $H^*(\widetilde{\mathbb{P}}^n, \mathbb{Z})$

We are going to make precise the multiplicative structure of the cohomology $H^*(\widetilde{\mathbb{P}}^n, \mathbb{Z})$. Since $\varphi^* : H^*(\widetilde{\mathbb{P}}^n, \mathbb{Z}) \longrightarrow H^*(\mathbb{P}^n, \mathbb{Z})$ is a ring homomorphism, it follows from Theorem in 5.

$$\xi_i \xi_j = \begin{cases} e_{ij}\xi_{i+j} & \text{if } i + j \leq n , \\ 0 & \text{if not} , \end{cases}$$

where $e_{ij} = l_i l_j / l_{i+j}$, $1 \leq i, j \leq n$.

In the polynomial ring $\mathbb{Z}[T_1, \cdots, T_n]$, let $\mathfrak{a}$ be the ideal generated by the elements

$$T_i T_j \ (i + j > n \text{ and } T_i T_j - e_{ij} T_{i+j} \ (i + j \leq n) .$$

We obtain a ring isomorphism

$$H^*(\widetilde{\mathbb{P}}^n, \mathbb{Z}) \simeq \mathbb{Z}[T_1, \cdots, T_n]/\mathfrak{a}$$

where ξ_i corresponds to the class of T_i.

7 The Generator ξ_1

Let us determine the homomorphism $\psi^* : H^*(\mathbb{P}^n, \mathbb{Z}) \longrightarrow H^*(\widetilde{\mathbb{P}}^n, \mathbb{Z})$ (1.(b)). Write $\psi = \psi(q)$, $\varphi = \varphi(q)$, $q = (q_0, \cdots, q_n)$, and make the identification $\mathbb{P}^n = \mathbb{P}^n(m, \cdots, m)$ where $m = l_1 = \operatorname{lcm}\{q_0, \cdots, q_n\}$. Then it holds $\psi(q) \circ \varphi(q) = \varphi(\underline{m})$. This implies $\psi^*(\xi) = \xi_1$ (Theorem in 5.), which determines completely ψ^*.

Since $\xi = c_1(L^*)$, it follows that $\xi_1 = c_1(\widetilde{L}^*)$, where $\widetilde{L} = \psi^*(L)$ is the canonical line bundle over $\widetilde{\mathbb{P}}^n$ (1.(b)). Denote $\widetilde{\xi} = c_1(\widetilde{L}^*)$ $(= \xi_1)$. From 6., we obtain

$$\xi^k = (l_1^k / l_k)\xi_k, \ 0 \leq k \leq n \ .$$

II. Étale Cohomology of Weighted Projective Spaces

We fix through all this chapter (II) a field k. Schemes, morphisms of schemes, will always mean k-schemes, k-morphisms of k-schemes ... For basic facts about schemes one may consult [M1], [H].

First of all, we want to define weighted projective spaces as quotients of $A^{n+1} \backslash \{0\}$ by the multiplicative group G_n, in the category schemes (A^{n+1} is the affine space, over k, and $G_m = GL(1)$).

1 Quotients of Schemes by Groups

A) Let X be a scheme and G be a group scheme acting on X [M2].

(a) DEFINITION. *A geometric quotient of X by G is a couple (Y, π), where Y is a scheme, $\pi : X \longrightarrow Y$ a morphism of schemes, such that (in the category of locally ringed spaces) the following sequence be exact*

$$(1)\quad G \underset{k}{\times} X \underset{p}{\overset{\sigma}{\rightrightarrows}} X \xrightarrow{\pi} Y ,$$

where σ stands for the action of G on X, and p is the second projection. This means that the next conditions hold.
$(QG)_1$ (i) $\pi \circ \sigma = \pi \circ p$,
(ii) if x, $x' \in X$ and $\pi(x) = \pi(x')$, then there exists $z \in G \times_k X$ such that $\sigma(z) = x$ and $p(z) = x'$,
(iii) π is onto;
$(QG)_2$ the topology of Y is the quotient topology defined by X and π;
$(QG)_3$ if $f = \pi \circ \sigma = \pi \circ p$, then the sequence of sheaves (given by (1))

$$(2)\quad 0 \longrightarrow O_Y \longrightarrow \pi_* O_X \rightrightarrows f_* O_{G \times X}$$

is exact. [2]

The kernel of the double arrow in the sequence (2) is, by definition, the subsheaf of invariants of $\pi_* O_X$ under G and is denoted $(\pi_* O_X)^G$. The condition $(QG)_3$ says that the structural sheaf of the scheme Y is $(\pi_* O_X)^G$.

The next proposition justifies the preceding definition.

Proposition ([M2]0, §2, 0.1)
A geometric quotient (Y, π) of X by G is a (categorical) quotient of X by G, i.e. for any commutative diagram (in the category of schemes)

---

[2] In this definition, the condition $(QG)_2$ is weaker than the corresponding one in the first edition of Geometric Invariant Theory, by D. Mumford (Chap.0, §1, 0.6).

$$\begin{array}{ccc} G\times_k X & \xrightarrow{\sigma} & X \\ p\downarrow & & \downarrow h \\ X & \xrightarrow[h]{} & Z \end{array},$$

there is a unique morphism $\bar{h} : Y \longrightarrow Z$ *with* $\bar{h} \circ \pi = h$*:*

$$X \xrightarrow[\pi]{} Y \xrightarrow[\bar{h}]{} Z \quad (h = \bar{h}\circ\pi : X \longrightarrow Z).$$

Hence a geometric quotient is unique up to isomorphism.

(b) Existence of a geometric quotient in the affine case

Assume the scheme X is affine:

$$X = \text{Spec } (A) \;, \quad \text{where } A \text{ is a } k\text{-algebra.}$$

Then the operation $\sigma : G \times_k X \longrightarrow X$ induces a k-algebra morphism

$$f : A \longrightarrow R \otimes_k A \;, \quad \text{where } R = \Gamma(G, O_G) \;.$$

An element $a \in A$ is invariant under G if $f(a) = 1 \otimes a$. The set A^G of invariant elements is a sub-k-algebra of A, and the inclusion $A^G \subset A$ defines a morphism of schemes $\pi : X \longrightarrow Y = \text{Spec } (A^G)$.

Theorem ([M2] 1, §2, 1.1 and 1.3)
If G is a linearly reductive algebraic group (over k), and if the orbits of geometric points of X are closed, then (Y, π) is a geometric quotient of X by G. Moreover, if X is noetherian (resp. algebraic), then Y is noetherian (resp. algebraic).

Recall that *linearly reductive* means linear and that every representation $G \longrightarrow GL(n)$ is completely reducible (cf. [M2] 1, §1).

B) Let now X be a scheme and G be a finite abstract group, acting on X by automorphisms. That is, we have a group homomorphism $G \longrightarrow \text{Aut}_k(X)$. This is equivalent to the datum of an action $G_k \times X \longrightarrow X$, where G_k is the constant group scheme associated to G. A *geometric quotient* of X by G is a geometric quotient of X by G_k (in the sense of A.(a)).

Assume X is affine: $X = \text{Spec }(A)$ (A k-algebra). Then $\text{Aut}_k(X) = \text{Aut}_k(A)$, and let $A^G \subset A$ denote the subalgebra of invariants (Notice that $A^G = A^{G_k}$, A^{G_k} being defined in A.(b)). This induces a morphism $\pi : X \longrightarrow Y = \text{Spec }(A^G)$.

From [G3] Exp. V, 1.1, 1.2, 1.5, 1.8, we have the following theorem about the existence of a geometric quotient of a scheme by the group G.

Theorem
i) (Y, π) is a geometric quotient of X by G. If X is algebraic, then so is Y.
ii) If X' is a scheme on which G acts, then a geometric quotient of X' by G exists if, and only if, X' is covered by affine open subsets which are stable under G.

2 Weighted Projective Spaces as Geometric Quotients

Remember that a ground field k has been fixed.

Let $q_0, \cdots, q_n$ be positive integers. Set

$$X = A^{n+1} \backslash \{0\} = \text{Spec }(R) \backslash \text{Spec }(k) ,$$

where $R = k[T_0, \cdots, T_n]$ is the polynomial k-algebra with $n+1$ indeterminates. Denote G the multiplicative group over k

$$G_m = \text{Spec }(k, [T, T^{-1}]) .$$

(a) Action of G on X

Consider the localizations

$$R_i = R_{T_i} , \ R_{ij} = R_{T_iT_j} \quad (0 \leq i, j \leq n) ,$$

and the associated affine schemes

$$X_i = \text{Spec } (R_i) \ , \ X_{ij} = \text{Spec } (R_{ij}) \ .$$

There are canonical k-algebra isomorphisms $R_{ij} \simeq R_{ji}$, inducing isomorphisms $X_{ji} \simeq X_{ij}$. We shall view X as the scheme obtained by gluing the X_i's along these isomorphisms between the open subsets $X_{ij} \subset X_i$.

There are k-algebra homomorphisms

$$R_i \longrightarrow k[T, T^{-1}] \otimes_k R_i \ , \ R_{ij} \longrightarrow k[T, T^{-1}] \otimes_k R_{ij}$$

defined by $T_s \longrightarrow T^{q_s} \otimes T_s$ $(0 \leq s \leq n)$. This gives actions of the group G on X_i and X_{ij}, let us say σ_i and σ_{ij}, such that the open immersions $X_{ij} \subset X_i$ are G-equivariant.

The operations σ_i's $(0 \leq i \leq n)$ glue together along the natural isomorphisms $\sigma_{ij} \simeq \sigma_{ji}$, into an operation $\sigma : G \times_k X \longrightarrow X$ of G on X. To make precise that this action depends on the integers $q_0, \cdots, q_n$, we shall write $\sigma = \sigma(q_0, \cdots, q_n)$.

(b) Quotient of X by G

For each i, $0 \leq i \leq n$, put $Y_i = \text{Spec } (R_i^G)$ and $\pi_i : X_i \longrightarrow Y_i$ the morphism induced by the inclusion $R_i^G \subset R_i$.

Lemma *The couple (Y_i, π_i) is a geometric quotient of X_i by G.*

Proof. Since G is linearly reductive, and because of Theorem 1.(b), we need only to show that the orbits of geometric points of X_i are closed. So take a geometric point $\epsilon : \text{Spec } (K) \longrightarrow X_i$, where K is an algebraically closed extension of k. Let $u : R_i \longrightarrow K$ be the corresponding k-homomorphism. We have to prove the image of the morphism

$$(\sigma_i \circ (1 \times \epsilon), p) : G \times_k \text{Spec } (K) \longrightarrow X_i \times_k \text{Spec } (K)$$

is closed (p is the second projection $G \times_k \text{Spec } (K) \longrightarrow \text{Spec } (K)$). But this corresponds to the k-homomorpishm

$$h : K[T_0, \cdots, T_n, T_i^{-1}] = R_i \otimes_k K \longrightarrow k[T, T^{-1}] \otimes_k K = K[T, T^{-1}] \ ,$$

defined by $h(T_s) = u(T_s)T^{q_s}$ $(0 \leq s \leq n)$. This h is integral, indeed:

$$T^{q_i} = u(T_i^{-1})(u(T_i)\,T^{q_i}) = u(T_i^{-1})h(T_i) = h(u(T_i^{-1})\,T_i)$$

and

$$T^{-q_i} = h(u(T_i)\,T_i^{-1})\ .$$

Therefore $(\sigma_i \circ (1 \times \epsilon), p)$ is closed, and so the orbit of ϵ is closed as required.

Similarly to the geometric quotients $\pi_i : X_i \longrightarrow Y_i$, there are geometric quotients $\pi_{ij} : X_{ij} \longrightarrow Y_{ij}$ $(o \leq i,j \leq n)$ in such a why that there exist open immersions $Y_{ij} \subset Y_i$ and isomorphisms $Y_{ij} \simeq Y_{ji}$ (use $(QG)_2$ and Proposition in 1.(a)). This gives the following commutative diagrams

$$\begin{array}{ccccccc} X_i & \hookrightarrow & X_{ij} & \simeq & X_{ji} & \hookleftarrow & X_j \\ \pi_i\downarrow & & \pi_{ij}\downarrow & & \downarrow\pi_{ji} & & \downarrow\pi_j \\ Y_i & \hookrightarrow & Y_{ij} & \simeq & Y_{ji} & \hookleftarrow & Y_j \end{array}$$

Gluing the π_i's along the isomorphisms $\pi_{ij} \simeq \pi_{ji}$, we obtain a morphism of schemes $\pi : X \longrightarrow Y$. So (Y, π) is a geometric quotient of X by G (since this property is local on Y; see Definition 1.(a) above).

DEFINITION. *The geometric quotient* (Y, π) *we just constructed is called* weighted (or twisted) projective space *of type* $(q_0, \cdots, q_n)$ *and is denoted*

$$\widetilde{\mathbb{P}}^n_k = \mathbb{P}_k(q_0, .., q_n)$$

(k is any field, $q_0, \cdots, q_n$ are any positive integers).

(c) The k-algebra R_i^G

By construction, the scheme $\widetilde{\mathbb{P}}^n$ is covered by open subsets

$$Y_i = Y_i(q_0, \cdots, q_n) = \text{Spec}\ (R_i^G) \quad (0 \leq i \leq n)\ .$$

Let us determine the algebra of invariants R_i^G. The action $\sigma_i : G \times_k X_i \longrightarrow X_i$ is defined by the k-algebra homomorphism

$$f_i : R_i = k[T_0, \cdots, T_n, T_i^{-1}] \rightarrow k[T, T^{-1}] \otimes_k R_i = k[T, T^{-1}, T_0, \cdots, T_n, T_i^{-1}],$$

$f_i(T_s) = T^{q_s}T_s \quad (0 \le s \le n)$.

An element of R_i of the form

$$a_{\underline{\alpha}} = T_0^{\alpha_0} \cdots T_n^{\alpha_n} \text{ where } \alpha_s \in \mathbb{N}\ (s \neq i),\ \alpha_i \in \mathbb{Z}\ ,$$

is invariant under G if, and only if,

$$T^{\beta} a_{\underline{\alpha}} = f_i(a_{\underline{\alpha}}) = a_{\underline{\alpha}}\ , \quad \text{with } \beta = \sum_0^n \alpha_s q_s\ ,$$

which is equivalent to saying $\beta = 0$. Thus the k-algebra of invariants R_i^G is generated by the elements $a_{\underline{\alpha}}$ above which verify $\sum_0^n \alpha_s q_s = 0$.

(d) REMARKS

i) Our construction of $\widetilde{\mathbb{P}}^n = \mathbb{P}_k(q_0, \cdots, q_n)$ works for all integers $q_0, \cdots, q_n \in \mathbb{Z}\backslash\{0\}$. However we keep the hypothesis $q_i > 0$ $(0 \le i \le n)$.

ii) Consider the $\mathbb{N}$-graduation on the k-algebra $R = k[T_0, \cdots, T_n]$ given by

$$\deg\,(T_s) = q_s\ (0 \le s \le n)\ ,\ \deg\,(\lambda) = 0\ (\lambda \in k)\ .$$

We shall write $R = R(q_0, \cdots, q_n)$ to specify this graduation. This induces a $\mathbb{Z}$-graduation on $R_i = R_{T_i}$, the 0-degree component of which is $R_{i,0} = R_i^G$ (after (c) above). Since the scheme $\mathrm{Proj}(R(q_0, \cdots, q_n))$ ([GD] II, §2) is constructed by gluing the affine schemes $\mathrm{Spec}(R_i, 0)$ $(0 \le i \le n)$, we obtain a canonical isomorphism

$$\mathbb{P}_k^n(q_0, \cdots, q_n) = \mathrm{Proj}\ (R(q_0, \cdots, q_n))\ .$$

In particular, it follows that $\mathbb{P}_k^n(1, \cdots, 1) = \mathbb{P}_k^n$ is the usual projective space over k.

iii) Let $q'_0, \cdots, q'_m$ be integers > 0.

(α)Assume $m \ge n$ and q'_i divides q_i $(0 \le i \le n)$. Let $V = \cup_{i=0}^n Y'_i$, where $Y'_i = Y_i(q'_0, \cdots, q'_n)$ is the affine open subset of $\mathbb{P}(q'_0, \cdots, q'_m)$ defined above (see (b) and (c)). It is clear that there is a morphism $V \longrightarrow \widetilde{\mathbb{P}}^n = \mathbb{P}(q_0, \cdots, q_n)$, induced by the k-algebra homomorphism

$$R(q_0, .., q_n) \longrightarrow R(q'_0, \cdots, q'_n)\ ,\ \ T_s \mapsto T_s^{\alpha_s}\ (0 \le s \le n)\ ,$$

where $\alpha_s = q_s/q'_s$ (glue the corresponding morphisms $Y'_i \longrightarrow Y_i$, $0 \leq i \leq n$).
(β) Assume now $m \leq n$ and q'_i divides q_i ($0 \leq i \leq n$). Then the k-algebra homomorphism

$$\begin{array}{rcl} k[T_0,\cdots,T_n] & \longrightarrow & k[T_0,\cdots,T_m] \\ T_s & \mapsto & T_s^{\alpha_s}\ (0 \leq s \leq m),\ \alpha_s = q_s/q'_s \\ T_s & \mapsto & 0\ (m+1 \leq s \leq n) \end{array}$$

induces a morphism $\mathbb{P}(q'_0,\cdots,q'_m) \longrightarrow \mathbb{P}(q_0,\cdots,q_n)$, which is a closed immersion if $q'_i = q_i$ ($0 \leq i \leq m$).
iv) For any integer $d > 0$, there is a natural isomorphism $\mathbb{P}(dq_0,\cdots,dq_n) = \mathbb{P}(q_0,\cdots,q_n)$. Indeed, consider the endomorphism u of G defined by $T \mapsto T^d$. Then $\sigma' = \sigma \circ (u \times \mathrm{id}_X)$, where σ and σ' are the actions of G defining the quotients $\mathbb{P} = \mathbb{P}(q_i)$ and $\mathbb{P}' = \mathbb{P}(dq_i)$ (see 2.(a)). This induces a morphism $\mathbb{P}' \longrightarrow \mathbb{P}$, which is an isomorphism since locally σ and σ' have the same invariants (by 2.(c)).

3 Some Properties of the Scheme $\widetilde{\mathbb{P}}^n_k$

(a) The scheme $\widetilde{\mathbb{P}}^n = \mathbb{P}_k(q_0,..,q_n)$ is integral and algebraic over k

It is integral since it is the quotient of $X = A_k^{n+1}\backslash\{0\}$ (by $G = G_n$) (after 2. and Proposition in 1.A.), and X is integral. Now it is algebraic since $\widetilde{\mathbb{P}}^n = \cup_{i=0}^n Y_i$, where $Y_i = Y_i(q_0,\cdots,q_n) \subset \widetilde{\mathbb{P}}^n$ are open immersions, and the Y_i's are algebraic (cf. 2.(c)).

(b) Proposition *The k-scheme $\widetilde{\mathbb{P}}^n$ is projective. More precisely there is a closed immersion $\widetilde{\mathbb{P}}^n \subset \mathbb{P}^m$ where the integer m is defined as follows. Let s be the smallest integer which is a multiple of $l = lcm\{q_0,\cdots,q_n\}$ and greater or equal to the sum $\sum_0^n (l - q_i)$. Let f be the complex function*

$$f(z) = [z^{s+1}(1 - z^{q_0} \cdots (1 - z^{q_n})]^{-1}\ .$$

Then $m = res_0(f) - 1$ (res = residue).

Proof. Consider the graded algebra $R = R(q_0, \cdots, q_n)$ defined in 2.(d).ii. For any integer $r \geq 1$, $R^{(r)}$ denotes the subalgebra generated by homogeneous elements of degree multiple of r. There is a graduation on $R^{(r)}$ in such a way that $(R^{(r)})_\alpha = R_{r\alpha}$. The integer s is defined in the statement above, the algebra $R^{(s)}$ is generated by its elements of degree 1 ([GD] II.2.1.6, ii) and v); in the proof of ii) the integer n_1 is equal to $\sum_0^n (l - q_i)$ in our situation). The monomials $T_0^{\alpha_0} \cdots T_n^{\alpha_n}$, of degree $s = \alpha_0 q_1 + \cdots + \alpha_n q_n$ in $R = k[T_0, \cdots, T_n]$ (i.e. of degree 1 in $R^{(s)}$) form a set the cardinality of which is $\mathrm{res}_0(f) = m + 1$. Let $a_0, \cdots, a_m$ be homogeneous elements of degree 1, generating the k-algebra $R^{(s)}$. So there is a graded k-algebra epimorphism

$$S = k[t_0, \cdots, t_m] \longrightarrow R^{(s)} : t_i \mapsto a_i \; (0 \leq i \leq m) ,$$

where S is the usual graded polynomial algebra. This induces a closed immersion ([GD] II.2.9.2.i).)

$$\mathrm{Proj}\ (R^{(s)}) \longrightarrow \mathrm{Proj}\ (S) = \mathbb{P}_k^m .$$

But $\mathrm{Proj}(R^{(s)}) = \mathrm{Proj}(R)$ (loc. cit. 2.4.7 i).) and $\mathrm{Proj}(R) = \widetilde{\mathbb{P}}_k^m$ (2.(d).ii).

(c) Local structure of $\widetilde{\mathbb{P}}^n$

Let q be an integer ≥ 1, and let $U_q = U_q(k)$ be the subgroup of the q-th roots of unity, in the multiplicative group $k^* = k \backslash \{0\}$. Denote by d the order of U_q (if k is algebraically closed and char (k) if 0 or is prime with q, then $d = q$). The constant algebraic group (over k), defined by the abstract group U_q, is the group of the d-th roots of unity:

$$\mu_d = \mathrm{Spec}\ (k[T]/(T^d - 1)) \ (\text{since } T^d - 1 = \prod_{\lambda \in U_q} (T - \lambda)) .$$

In particular, an action of U_q (by automorhisms) on a scheme S is equivalent to an action of μ_d on S.

We have $\widetilde{\mathbb{P}}^n = \cup_0^n Y_i$, where $Y_i = Y_i(q_0, \cdots, q_n)$ is an open subset (see 2.(c)), complementary to the closed immersion

$$\mathbb{P}^{n-1}(q_0, \cdots, \hat{q}_i, \cdots, q_n) \subset \widetilde{\mathbb{P}}^n = \mathbb{P}^n(q_0, \cdots, q_n)$$

defined by the projection

$$k[T_0,\cdots,T_n] \longrightarrow k[T_0,\cdots,\hat{T}_i,\cdots,T_n] : T_j \mapsto (1-\delta_{ij})T_j \; .$$

If the order of U_{q_i} is q_i, then Y_i is a geometric quotient of the affine space $A^n = \text{Spec}\ (k[T_0,\cdots,\hat{T}_i,\cdots,T_n])$ by U_{q_i}. In this case we shall write

$$Y_i = A^n/U_{q_i} = (A^n/U_{q_i})(q_0,\cdots,\hat{q}_i,\cdots,q_n) \; .$$

Proof. To simplify the notation, we assume $i = 0$, and put $q = q_0$. The action of U_q on $A^n = \text{Spec}\ (k[T_1,\cdots,T_n])$ is defined by the group homomorphism

$$U_q \longrightarrow \text{Aut}_k(k[T1,\cdots,T_n]) : \lambda \mapsto u_\lambda \; , \; u\lambda(T_s) = \lambda^{q_s} T_s \; .$$

Let B be the associated subalgebra of invariants. Since $Y_0 = \text{Spec}(R_0^G)$ (2.(c)), it is enough to verify that the two algebras B and R_0^G are isomorphic. But B is generated by the monomials $T_1^{\alpha_1}\cdots T_n^{\alpha_n}$ with $\sum_{s=1}^{n} \alpha_s q_s \equiv 0 \bmod |U_q|$. Using 2.(c), we have the following homomorphism

$$\begin{array}{ccccccl} B & \rightarrow & R_0^G & : & T_1^{\alpha_1}\cdots T_n^{\alpha_n} & \mapsto & T_0^{\alpha_0}\cdots T_n^{\alpha_n} \text{ with } \alpha_0 = -(\sum_1^n \alpha_s q_s)/q_0 \; , \\ R_0^G & \rightarrow & B & : & T_0^{\alpha_0}\cdots T_n^{\alpha_n} & \mapsto & T_1^{\alpha_1}\cdots T_n^{\alpha_n} (\text{where } \sum_0^n \alpha_s q_s = 0) \; , \end{array}$$

which are inverse each to the other.

(d) Morphisms between $\widetilde{\mathbb{P}}^n$ and $\mathbb{P}^n$

There are two morphisms

$$\begin{array}{rcl} \varphi = \varphi(q_0,\cdots,q_n) & : & \mathbb{P}^n \longrightarrow \widetilde{\mathbb{P}}^n = \mathbb{P}(q_0,\cdots,q_n) \\ \psi = \psi(q_0,\cdots,q_n) & : & \widetilde{\mathbb{P}}^n \longrightarrow \mathbb{P}^n = \mathbb{P}(1,\cdots,1) \; , \end{array}$$

defined as follows. The first one is induced by the graded k-algebra homomorphism (2.(d) ii.)

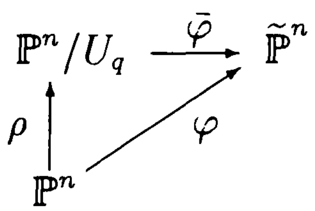
$$R(q_0,\cdots,q_n) \longrightarrow R(1,\cdots,1) : T_i \mapsto T_i^{q_i} \; (0 \le i \le n) \; .$$

To define the second one, let $l = \text{lcm}\{q_0, \cdots, q_n\}$ and put $s_i = l/q_i$. Then the graded k-algebra homomorphism

$$R(l, \cdots, l) \longrightarrow R(q_0, \cdots, q_n) : T_i \mapsto T_i^{s_i} \; (0 \leq i \leq n)$$

induces a morphism $\widetilde{\mathbb{P}}^n \longrightarrow \mathbb{P}^n(l, \cdots, l)$ which, composed with the natural isomorphism $\mathbb{P}^n(l, \cdots, l) \xrightarrow{\sim} \mathbb{P}^n$ (2.(d) iv.), gives ψ.

Proposition *Let* $q = (q_0, \cdots, q_n)$ *and* $U_q = U_{q_0} \times \cdots \times U_{q_n}$. *Then the group* U_q *acts (by automorphisms) on* $\mathbb{P}^n$ *in such a way that a geometric quotient* $(\mathbb{P}^n/U_q, \rho)$ *exist with a commutative diagram*



If $|U_{q_i}| = q_i$ *for every* i, *then* $\bar{\varphi}$ *is an isomorphism.*

Proof. i) Action of U_q on $\mathbb{P}^n$: for $\lambda = (\lambda_0, \cdots, \lambda_n) \in U_q$, let $h(\lambda)$ be the automorphism of $\mathbb{P}^n$ defined by $T_i \longrightarrow \lambda_i T_i$ in the usual graded algebra $R(1) = k[T_0, \cdots, T_n]$ (2.(d) ii). We obtain a group homomorphism $h : U_q \longrightarrow \text{Aut}_k(\mathbb{P}^n)$, that is an action of U_q on $\mathbb{P}^n$. The standard affine open subsets $Y_i(1)$ $(0 \leq i \leq n)$ are stable under U_q. Therefore there exists a geometric quotient $(\mathbb{P}^n/U_q, \rho)$ (cf. Theorem in 1.B).

ii) Let us recall some notation: $R = k[T_0, \cdots, T_n]$, $R_i = R_{T_i}$; $R(q) = R(q_0, \cdots, q_n)$ means that R is graded by deg $(T_i) = q_i$; $Y_i(q) =$ Spec $(R_i^G(q))$ where G is the multiplicative group (see 2.(c) and 2.(d) ii). Now we construct the morphism $\bar{\varphi}$.

Let $B_i \subset R_i^G(1) = k[\frac{T_0}{T_i}, \cdots, \frac{T_n}{T_i}]$ denote the subalgebra of invariants under U_q. So we have (Theorem in 1.B)

$$\rho(Y_i(1)) = Y_i(1)/U_q = \text{Spec }(B_i) \; .$$

On the other hand, there are commutative diagrams of k-algebras

$$\begin{array}{ccc} R_i^G(q) & \xrightarrow{u} & R_i^G(1) \\ {\scriptstyle v}\uparrow & \nearrow{\scriptstyle \text{inclusion}} & \\ B_i & & \end{array}$$

such that
$$\begin{aligned} u(T_0^{\alpha_0}\cdots T_n^{\alpha_n}) &= T_0^{\alpha_0 q_0}\cdots T_n^{\alpha_n q_n}\ , \\ v(T_0^{\alpha_0}\cdots T_n^{\alpha_n}) &= T_0^{\alpha_0 q_0}\cdots T_n^{\alpha_n q_n}\ : \end{aligned}$$

recall that $T_0^{\alpha_0}\cdots T_n^{\alpha_n} \in R_i^G(q)$ if and only if $\sum_0^n \alpha_s q_s = 0$ and that such "monomials" generate $R_i^G(q)$. Note also that u is induced by the homomorphism $R(q) \longrightarrow R(1)$ defining φ. Now, if $|U_{q_i}| = q_i$ $(0 \leq i \leq n)$, then v is an isomorphism. Indeed, in that case, an element of $R_i^G(1)$ of the form

$$a_{\underline{\alpha}} = T_0^{\alpha_0}\cdots T_n^{\alpha_n}\ , \quad \text{where } \alpha_s \in \mathbb{N}(s \neq i),\ \alpha_i \in \mathbb{Z},\ \text{and } \sum_0^n \alpha_s = 0\ ,$$

is invariant under U_q if, and only if,

$$\lambda_0^{\alpha_0}\cdots\lambda_n^{\alpha_n} a_{\underline{\alpha}} = a_{\underline{\alpha}} \text{ for all } \lambda = (\lambda_0,\cdots,\lambda_n) \in U_q\ ,$$

that is if, and only if, q_s divides α_s for all s.

The diagrams above define commutative diagrams

$$\begin{array}{ccc} Y_i(1) & \longrightarrow & Y_i(q) \\ & \searrow & \uparrow \\ & & Y_i(1)/U_q \end{array}$$

which, when glued together, give the required diagram.

(e) Canonical sheaves over $\widetilde{\mathbb{P}}^n$

The canonical sheaves $O_X(s)$ over $X = \widetilde{\mathbb{P}}^n$ are defined by the graded $R(q)$-modules $R(q)[s]$. By definition of the morphism $\psi : X = \widetilde{\mathbb{P}}^n \longrightarrow Y = \mathbb{P}^n$, we see that $\psi^* O_Y(r) = O_X(lr)$, where $l = \text{lcm}\{q_0,\cdots,q_n\}$. In particular the sheaf $O_X(l)$ is invertible, and is considered as the canonical invertible sheaf over $\widetilde{\mathbb{P}}^n$. Modulo a normalization of the degrees $q_0,\cdots,q_n$, it is shown in [A2] that $O_X(s)$ is invertible if, and only if, l divides s; moreover Pic $(\widetilde{\mathbb{P}}^n) = \mathbb{Z}$ with generator $O_X(l)$.

4 Degree of the Morphism φ

In this section, we assume the ground field k is algebraically closed and the integers $q_0, .., q_n$ are prime to the characteristic exponent of k (equal to 1 if char $(k) = 0$ and equal to char (k) if not). Let a be an integer ≥ 1, prime to the characteristic exponent of k, and put $F = \mathbb{Z}/a\mathbb{Z}$. We want to compute the étale cohomology homomorphism $\varphi^* : H^{2n}(\widetilde{\mathbb{P}}^n, F_{\widetilde{\mathbb{P}}}) \longrightarrow H^{2n}(\mathbb{P}, F_{\mathbb{P}})$, induced by $\varphi : \mathbb{P} = \mathbb{P}_k^n \longrightarrow \widetilde{\mathbb{P}} = \mathbb{P}_k^n(q_0, \cdots, q_n)$.

(a) First let us see that $\widetilde{\mathbb{P}}$ has the form $\mathbb{P}/\bar{H}$, where $\bar{H}$ is a finite abstract group acting faithfully on $\mathbb{P}$.

i) A finite abstract group H, acting on a scheme S (by automorhisms), acts faithfully on S if the homomorphism $H \longrightarrow$ Aut (S), defining the action of H on S, is injective. Consider an action of H on S and put $H_0 =$ Ker $(H \longrightarrow$ Aut $(S))$. Then $\bar{H} = H/H_0$ acts on S faithfully and, for every (geometric) quotient $(S/H, \rho)$ there is a (geometric) quotient $(S/\bar{H}, \bar{\rho})$ with a commutative diagram (see Theorem in 1.B)

$$\begin{array}{ccc} & S & \\ {}^{\rho}\swarrow & & \searrow^{\bar{\rho}} \\ S/H & \xrightarrow{\sim} & S/\bar{H} \end{array} .$$

ii) In the situation of Proposition (3.(d)), it is immediate that $H_0 = U_d$ with $d = \gcd\{q_0, \cdots, q_n\}$ $(S = \mathbb{P}^n, H = U_q)$. Since $|U_{q_i}| = q_i$ (by hypotheses above), we have

$$\widetilde{\mathbb{P}}^n = \mathbb{P}^n/\bar{H} \text{ with } \bar{H} = U_q/U_d ,$$

and $|\bar{H}| = q_0 \cdots q_n/d$. Therefore the homomorphism $\varphi^* : H^{2n}(\widetilde{\mathbb{P}}^n, F_{\widetilde{\mathbb{P}}}) \longrightarrow H^{2n}(\mathbb{P}, F_{\mathbb{P}})$ we want to determine has the form

$$\bar{\rho}^* : H^{2n}(\mathbb{P}^n, F_{\bar{\mathbb{P}}}) \longrightarrow H^{2n}(\mathbb{P}, F_{\mathbb{P}}) \text{ where } \bar{\rho} : \mathbb{P} \longrightarrow \mathbb{P}/\bar{H} .$$

The following theorem shows that this is nothing else but multiplication by the integer $|\bar{H}| = q_0 \cdots q_n/d$ in the group F.

(b) Theorem *Let S be an integral, smooth scheme, of finite type over k and of dimension n. Let H be a finite group acting faithfully on S such that a geometric quotient $(S/H, \rho)$ of S by H exist. Then we have*

$$\text{(i) } H_c^{2n}(S/H, F_{S/H}) = F \ ,$$
$$\text{(ii) } (H_c^{2n}(S/H, F_{S/H}) \xrightarrow{\rho^*} H_c^{2n}(S, F_s)) = (F \xrightarrow{\cdot|H|} F) \ ,$$

where $H_c^{\cdot}$ stands for cohomology groups with compact supports, [DV].

Proof. We may assume $H \neq \{1\}$. So let $g \in H$, $g \neq 1$, and denote by S^g the closed subscheme of invariants of S by the subgroup generated by g (By hypothesis a geometric quotient of S by H exist, hence (Theorem 1.B) S is covered by affine open subsets $V_\alpha = \text{Spec }(A_\alpha)$ which are stable under H. Let $\mathfrak{a}_\alpha^g$ be the ideal of A_α generated by $gx - x$, $x \in A_\alpha$; then $S^g \cap V_\alpha = \text{Spec }(A_\alpha/\mathfrak{a}_\alpha^g)$.) The action of H on S is faithful, so $S^g \neq S$ (S is reduced). Hence dim $(S^g) \leq n - 1$, and so dim $(T) \leq n - 1$ with $T = \cup_{g\neq 1} S^g$. On the other hand, the open subset $V = S\backslash T$ is non-empty, stable under H, and the action of H on it has no inertia. Thus we have a principal H-covering $\rho|_V : V \longrightarrow V/H$. Now the closed subscheme T of S is invariant by H and covered by stable affine open subsets ($T = \cup_\alpha \text{Spec }(B_\alpha)$, $B_\alpha = A_\alpha / \cap_{g\neq 1} \mathfrak{a}_\alpha^g$). Therefore a geometric quotient $(T/H, \gamma)$ exists so that T/H is a closed subscheme of S/H, complementary to V/H. This shows that we have an exact sequence morphism (coefficients are in $F = \mathbb{Z}/a\mathbb{Z}$)

$$\begin{array}{ccccccc}
0 = H_c^{2n-1}(T) & \to & H_c^{2n}(V) & \to & H_c^{2n}(S) & \to & H_c^{2n}(T) = 0 \\
\uparrow \gamma^* & & \uparrow \rho|_V^* & & \uparrow \rho^* & & \uparrow \gamma^* \\
0 = H_c^{2n-1}(T/H) & \to & H_c^{2n}(V/H) & \to & H_c^{2n}(S/H) & \to & H_c^{2n}(T/H) = 0
\end{array}$$

($\rho^*, \gamma^*, \cdots$ are well-defined since $\rho, \gamma, \cdots$ are proper!). It follows that:
1) $H_c^{2n}(S/H) = H_c^{2n}(V/H) = F$ (V/H is irreducible since V is),
2) to see (ii), it is sufficient to establish

$$(H_c^{2n}(V/H) \xrightarrow{\rho|_V^*} H_c^{2n}(V)) = (F \xrightarrow{\cdot|H|} F) \ .$$

Set $\tau = \rho|_V$ and $\bar{V} = V/h$. There is a commutative triangle

$$\begin{array}{ccc} H_c^{2n}(V) & \xrightarrow{\tau_*} & H_c^{2n}(\bar{V}) \\ & {\scriptstyle \mathrm{Tr}_V} \searrow^{\sim} \quad {}^{\sim}\swarrow {\scriptstyle \mathrm{Tr}_{\bar{V}}} & \\ & F & \end{array} \qquad (\mathrm{Tr} = \text{trace homomorphism}).$$

Hence it is enough to see that

$$\tau^* \tau_*(x) = |H| x, \text{ for all } x \in H_c^{2n}(V) \, .$$

But this is the case if $\tau^* \tau_*(1) = |H| 1$ in $H_c^0(V)$ because of the projection formula $\tau^* \tau_*(x) = x \tau^* \tau_*(1)$. Since $\tau : V \longrightarrow V/H$ is a principal H-covering, there is a commutative diagram

$$\begin{array}{ccc} H \times V = V \times_{\bar{V}} V & \xrightarrow{p_1} & V \\ \searrow \quad \downarrow {\scriptstyle p_2} & & \downarrow {\scriptstyle \tau} \\ V & \xrightarrow[\tau]{} & \bar{V} \end{array} \qquad (H \times V = \textstyle\coprod_H V)$$

Therefore $p_{1*} p_2^*(1) = |H| \cdot 1$ in $H_c^0(V)$. The proper change of basis theorem ([AR1] 5.1) says that $\tau^* \tau_*(1) = p_{1*} p_2^*(1)$, and this ends the proof.

5 Cohomology of a Cone

(a) Let q be an integer ≥ 1 and assume that $|U_q| = q$ (3.(c)). Then the quotient scheme $A^{n+1}/U_q = (A^{n+1}/U_q)(q_0, \cdots, q_n)$, defined in 3.(c), is the affine projecting cone of a projective variety (over k) the vertex of which is the image of $(0, \cdots, 0) \in A^{n+1}$.

Indeed the proof in 3.(c) shows that $A^{n+1}/U_q = \mathrm{Spec}(B)$, where B is the subalgebra of $R = k[T_0, \cdots, T_n]$ generated by the monomials

$$a_{\underline{\alpha}} = T_0^{\alpha_0} \cdots T_n^{\alpha_n} \text{ such that } q \text{ divides } \sum_0^n \alpha_s q_s \, .$$

Setting, for these elements, deg $(a_{\underline{\alpha}}) := (\sum_0^n \alpha_s q_s)/q$, we define an $\mathbb{N}$-graduation on B, that is an action of the line $D_k = \mathrm{Spec}(k[T])$

on A^{n+1}/U_q. Hence A^{n+1}/U_q is an affine cone with vertex the point $(0,\cdots,0)$. On the other hand, there is an integer $m \geq 1$ such that a closed immersion Proj $(B) \subset \mathbb{P}_K^N$ exist. This is because the k-algebra B of invariants is finitely generated (Theorem in 1.B), and, for each integer $r \geq 1$, the number of monomials $T_0^{\alpha_0}\cdots T_n^{\alpha_n}$ with $\sum_0^n \alpha_s q_s = rq$ is finite. Continuation of the proof is then similar to the one in 3.(b).

The following proposition shows that, if k is algebraically closed, then $H^0(A^{n+1}/U_q, F) = F$, $, H^i(A^{n+1}/U_q, F) = 0$ $(i \neq 0)$, and $H_0^i(A^{n+1}/U_q, F) \longrightarrow H_c^i(A^{n+1}/U_q, F)$ is an isomorphism for all i.

(b) Cohomology of a cone ([DE] 2.1.2, 2.1.3)[3]

We assume the ground field k algebraically closed.

Proposition *Let C be the affine projecting cone of a projective variety $V \subset \mathbb{P}_k^r$, and let 0 be the vertex of C. Let F be a torsion abelian group, prime to the characteristic exponent of k. Then we have:*
(i) $H^0(C,F) = F$, $H^i(C,F) = 0$ $(i \neq 0)$,
(ii) $H_0^i(C,F) \xrightarrow{\sim} H_c^i(C,F)$ $(i \geq 0)$.

Before the proof, we make a definition and establish a lemma.

Let S, T be two schemes (over k), A a torsion ring prime to the characteristic exponent of k, K an object in $D^+(S,A)$ and L an object in $D+(T,A)$ (derived category of A_T-Module complexes which are bounded below). A morphism $\varphi : (S,K) \longrightarrow (T,L)$ consists (by definition) of two morphisms $(S \xrightarrow{\varphi} T$, $\psi^* L \longrightarrow K)$. It induces $\psi^* : H^*(T,L) \longrightarrow H^*(S,K)$.

DEFINITION. *Two morphisms $\varphi_0, \varphi_1 : (S,K) \longrightarrow (T,L)$ are said to be homotopic if there exist a connected scheme Γ, of finite type over k, two points 0 and 1 in Γ and a morphism $\Phi : (S \times \Gamma, p_1^* K) \longrightarrow (T,L)$, where p_1 is the projection $S \times \Gamma \longrightarrow S$, with the following property. On the fibers over 0 and 1 of the projection $p_2 : S \times \Gamma \longrightarrow \Gamma, \Phi$ induces φ_0 and φ_1.*

[3] This paragraph is entirely borrowed from [DE] for our convenience.

Lemma *If $\varphi_0, \varphi_1 : (S,K) \longrightarrow (T,L)$ are homotopic, then $\varphi_0^* = \varphi_1^*$.*

Proof of Lemma. Keeping the same notation as in the definition above, there exist a sequence of points $0 = x_0, x_1, \cdots, x_m = 1$ in Γ, connected smooth curves Γ_i and morphisms $\gamma_i : \Gamma_i \longrightarrow \Gamma$ such that x_i and x_{i+1} belong to he image of γ_i (Γ_i is obtained by normalization of one-dimensional connected subscheme of Γ containing 0 and 1). Thus we may assume that Γ is a smooth curve. Now apply the smooth change of basis theorem ([AR2] 1.2) to the commutative square

$$\begin{array}{ccc} S \times \Gamma & \xrightarrow{p_1} & S \\ \beta \downarrow & & \downarrow f \\ \Gamma & \xrightarrow[g]{} & \mathrm{Spec}(k) \end{array}$$

Whence $g^* R^i f_* K \xrightarrow{\sim} R^i p_{2*}(p_1^* K)$, which gives the following commutative diagrams (with $x = 0, 1$)

$$\begin{array}{ccccc} H^i(T,L) & \xrightarrow{\varphi_x^*} & H^i(S,K) & \xrightarrow{\sim} & \\ \Phi^* \downarrow & & \uparrow \text{fiber over } x & & H^0(\Gamma, g^* H^i(S,K)) . \\ H^i(S \times \Gamma, p_1^* K) & \longrightarrow & H^0(\Gamma, R^i p_{2*}(p_1^* K)) & \xrightarrow{\sim} & \end{array}$$

But this means that φ_x^* is independent of x.

Proof of Proposition. (i) The projective variety $V \subset \mathbb{P}_k^r$ is defined by equations $f_1 = \cdots = f_s = 0$, where the f_i's are homogeneous polynomials in $r+1$ variables $x_0, \cdots, x_r$. The affine projecting cone $C \subset A_k^{r+1}$ of V is given by the same equations. The morphism $C \times A_k^1 \longrightarrow C$, defined by $((x_0, \cdots, x_r), t) \mapsto (tx_0, \cdots, tx_r)$, induces the identity of C at $t = 1$ and the constant morphism at $t = 0$ with value 0 (vertex of C). So the Lemma implies $H^i(C, F) = H^i(\{0\}, F)$.
(ii) The projective projecting cone $\hat{C} \subset \mathbb{P}_k^{r+1}$ of V is defined by the same equations $f_1 = \cdots = f_s = 0$ as in (i) except that the homogenous polynomials f_i are considered with $r+2$ variables $x_0, \cdots, x_{r+1}$. The morphism $(\hat{C} \backslash \{0\}) \times A_k^1 \longrightarrow \hat{C} \backslash \{0\}$, given by $(((x_0, \cdots, x_{r+1}, t) \mapsto (x_0, \cdots, x_r, tx_{r+1})$, shows that $V \subset \hat{C} \backslash \{0\}$ is a deformation re-

tract of $\hat{C}\backslash\{0\}$. Whence $H^i(\hat{C}\backslash\{0\})\xrightarrow{\sim}H^i(V)$ by Lemma above (coefficients are in F). The 5 lemma and the following commutative diagram give the required result.

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^{i-1}(\hat{C}\backslash\{0\}) & \longrightarrow & H^i_0(\hat{C}) & \longrightarrow & H^i(\hat{C}) & \longrightarrow & H^i(\hat{C}\backslash\{0\}) & \longrightarrow \cdots \\ & \downarrow\wr & & \downarrow & & \downarrow = & & \downarrow\wr & \\ \cdots \longrightarrow & H^{i-1}(V) & \longrightarrow & H^i_c(C) & \longrightarrow & H^i(\hat{C}) & \longrightarrow & H^i(V) & \longrightarrow \cdots \end{array}$$

$(H^i_0(\hat{C}) = H^i_0(C))$.

6 Étale Cohomology of $\tilde{\mathbb{P}}^n$

In all this section, the ground field k is algebraically closed, and the integers $q_0, \cdots, q_n$ are prime to its characteristic exponent. The weighted projective space $\mathbb{P}_k(q_0, \cdots, q_n)$ is denoted by $\tilde{\mathbb{P}}^n$.

Let a be an integer ≥ 1 and l a prime number, both prime to the characteristic exponent of k. Let $F = \mathbb{Z}/a\mathbb{Z}$ or $F = \mathbb{Z}_l$ (ring of l-adic integers) and S be a scheme. We write $H^i(S, F) = H^i(S, F_S)$ for the i-th étale ($F = \mathbb{Z}/a\mathbb{Z}$) or l-adic ($F = \mathbb{Z}_l$) cohomology group of the scheme S, with coefficients in the constant sheaf F_S defined by F. By definition one has

$$H^i(S, \mathbb{Z}_l) = \varprojlim_{\alpha \geq 1} H^i(S, \mathbb{Z}/l^\alpha\mathbb{Z}) .$$

A good reference to learn about étale cohomology (and other cohomologies of algebraic varieties) is DANILOV [DA].

(a) The groups $H^i(\tilde{\mathbb{P}}^n, F)$

Theorem *Let* $\tilde{\mathbb{P}}^n = \mathbb{P}_k(q_0, \cdots, q_n)$ *be the weighted projective space over* k, *of type* $(q_0, \cdots, q_n)$. *Then*
i) $H^i(\tilde{\mathbb{P}}^n, \mathbb{Z}/a\mathbb{Z}) = \mathbb{Z}/a\mathbb{Z}$ *if* $i = 2r$ $(0 \leq r \leq n)$, $= 0$ *if not;*
ii) $H^i(\tilde{\mathbb{P}}^n, \mathbb{Z}_l) = \mathbb{Z}_l$ *if* $i = 2r$ $(0 \leq r \leq n)$, $= 0$ *if not.*

Proof. Obviously (i) implies (ii). The proof of (i) is quite similar to the proof of Lemma 1 in I.4.(a) (in the Proposition in I.4.(a) take a proper scheme over k instead of a compact space).

(b) The homomorphism φ^*

Theorem *Let* $\varphi : \mathbb{P}^n \longrightarrow \widetilde{\mathbb{P}}^n$ *be the morphism defined in 3.(d), and* $l_i = l_i(q_0, \cdots, q_n)$ *the integers considered in I.5. For* $0 \leq i \leq n$, *we have:*

$$(H^{2i}(\widetilde{\mathbb{P}}^n, \mathbb{Z}/a\mathbb{Z}) \xrightarrow{\varphi^*} H^{2i}(\mathbb{P}^n, \mathbb{Z}/a\mathbb{Z})) = (\mathbb{Z}/a\mathbb{Z} \xrightarrow{\cdot l_i} \mathbb{Z}/a\mathbb{Z}) .$$

Proof. Because of 4.(a).ii, the proof is the same as that of the Theorem in I.5.

(c) Corollary *Let* q *be an integer* ≥ 1, *prime to the characteristic exponent of* k. *Consider the quotient scheme* $A^{n+1}/\mu_q = (A^{n+1}/U_q)(q_0, \cdots, q_n)$ *(cf. 3.(c)). Its cohomology with compact supports is as follows.*

$$\begin{aligned} H_c^u(A^{n+1}/\mu_q, \mathbb{Z}/a\mathbb{Z}) &= \mathbb{Z}/a\mathbb{Z} \text{ if } i = 2(n+1) , \\ &= \mathbb{Z}/(a, m_r)\mathbb{Z} \text{ if } i = 2r \text{ or } 2r+1,\ 0 \leq r \leq n, \\ &= 0 \text{ if not,} \end{aligned}$$

where $(a, m_r) = gcd\{a, m_r\}$, $m_r = l_r(q_0, \cdots, q_n, q)/l_r(q_0, \cdots, q_n)$ *(I.5).*

Proof. By 3.(c) and setting $\widetilde{\mathbb{P}}^{n+1} = \mathbb{P}(q_0, \cdots, q_n, q)$, A^{n+1}/μ_q is the complementary open subset of the closed immersion $\widetilde{\mathbb{P}}^n \subset \widetilde{\mathbb{P}}^{n+1}$. Thus we have the exact sequences (with coefficients in $\mathbb{Z}/a\mathbb{Z}$)

$$0 \to H_c^{2i}(A^{n+1}/\mu_q) \to H_c^{2i}(\widetilde{\mathbb{P}}^{n+1}) \to H^{2j}(\widetilde{\mathbb{P}}^n) \to H_c^{2j+1}(A^{n+1}/\mu_q) \to 0.$$

But the preceding theorem shows that the homomorphism $H^{2j}(\widetilde{\mathbb{P}}^{n+1}) \longrightarrow H^{2j}(\widetilde{\mathbb{P}}^n)$ is equal to multiplication by m_j in $\mathbb{Z}/a\mathbb{Z}$ $(0 \leq i \leq n)$.

(d) The ring $H^*(\widetilde{\mathbb{P}}^n, \mathbb{Z}_l)$

In the same manner and with the same notation as I.6, we have a ring isomorphism

$$H^*(\widetilde{\mathbb{P}}^n, \mathbb{Z}_l) \simeq \mathbb{Z}_l[T_1, \cdots, T_n]/\mathfrak{a} .$$

III. Cohomology of Weighted Projective Bundles

1 Definition of a Weighted Projective Bundle

The ground field in this chapter is the complex numbers $\mathbb{C}$. Let $q_0, \cdots, q_n$ be integers ≥ 1 and let $E = \bigoplus_0^n E_i$ be a direct sum of vector bundles E_i with constant rank ≥ 1, over a compact space X. The projection $E \longrightarrow X$ is denoted by π. The multiplicative group $\mathbb{C}^*$ acts on E as follows:

$$\sigma : \mathbb{C}^* \times E \longrightarrow E,\ \sigma(\lambda, (u_0, \cdots, u_n)) = (\lambda^{q_0} u_0, \cdots, \lambda^{q_n} u_n)$$

where $u_i \in E_i$. A motivation of considering such an action is given in Introduction.

Let $E^* = E \backslash X$ where $X \subset E$ is the null section of E. The topological quotient $E^*/\mathbb{C}^*$ of E^* modulo the operation σ will be denoted by

$$\widetilde{\mathbb{P}}(E) = \mathbb{P}(\bigoplus_0^n E_i\ ; q_0, \cdots, q_n)\ .$$

There is a projection $\rho : \widetilde{\mathbb{P}}(E) \longrightarrow X$, induced by $\pi : E \longrightarrow X$, the fibers of which are weighted projective spaces. More precisely, if $rk(E_i) = r_i$, $r = (r_0 + \cdots + r_n) - 1$, then any fiber of ρ is equal to the complex weighted projective space (cf. I.)

$$\widetilde{\mathbb{P}}^r = \mathbb{P}_{\mathbb{C}}(q_0, \cdots, q_n) \text{ with } q_i = (q_i, \cdots, q_i) \in \mathbb{N}^{r_i}\ .$$

The bundle $\rho : \widetilde{\mathbb{P}}(E) \longrightarrow X$ is called *weighted projective bundle.* If $n = 0$ or $q_0 = \cdots = q_n = 1$, then $\widetilde{\mathbb{P}}(E) = \mathbb{P}(E)$ is the usual projective bundle associated to the vector bundle E.

2 Canonical Line Bundle over $\widetilde{\mathbb{P}}(E)$

Let $q_{n+1} = \text{lcm}\{q_0, \cdots, q_n\}$ and $E_{n+1} = X \times \mathbb{C}$ (trivial line vector bundle over X). Consider then the weighted projective bundle $\widetilde{\mathbb{P}}(F)$ associated to $(F = \bigoplus_0^{n+1} E_i \ ; q_0, \cdots, q_{n+1})$. There is a closed immersion:

$$X = \mathbb{P}(E_{n+1}; q_{n+1}) \subset \mathbb{P}(\bigoplus_0^{n+1} E_i; q_0, \cdots, q_{n+1}) = \mathbb{P}(F) \ ,$$

defined by the canonical inclusion $E_{n+1} \subset \bigoplus_0^{n+1} E_i$. Denote $U = \widetilde{\mathbb{P}}(F) \backslash X$ its complementary open subset. Thus we have a projection $U \longrightarrow \widetilde{\mathbb{P}}(E)$ (given by $(u_0, \cdots, u_{n+1}) \mapsto (u_0, \cdots, u_n)$, $u_i \in E_i$), and this is a rank 1 vector bundle over $\widetilde{\mathbb{P}}(E)$ (the reason is that q_{n+1} is divisible by each q_i, $0 \leq i \leq n$; compare with I.1.(c)). We write $\widetilde{L}_E$ for this bundle and call it *the canonical line bundle* over $\widetilde{\mathbb{P}}(E)$.

More generally if, instead of q_{n+1}, we take any common multiple s of the integers q_i, we obtain a line bundle over $\widetilde{\mathbb{P}}(E)$ isomorphic to $(\widetilde{L})_E^{\otimes k}$ with $k = s/q_{n+1}$.

The restriction of $(\widetilde{L})_E$ to the fiber $\widetilde{\mathbb{P}}^r$ of $\widetilde{\mathbb{P}}(E)$ is the canonical line bundle defined in I.1.(b).

When Y is a topological space, then $H^*(Y)$ will stand for the integral cohomology $H^*(Y, \mathbb{Z})$. The class of $\widetilde{L}_E^*$ in the cohomology group $H^2(\widetilde{\mathbb{P}}(E))$ is denoted by $\widetilde{\xi}_E$.

There is a structure of $H^*(X)$-module on $H^*(\widetilde{\mathbb{P}}(E)) = \bigoplus_{i \geq 0} H^i(\widetilde{\mathbb{P}}(E))$ via the homomorphism $\rho^* : H^*(X) \longrightarrow H^*(\widetilde{\mathbb{P}}(E))$, induced by the projection $\rho : \widetilde{\mathbb{P}}(E) \longrightarrow X$.

3 Cohomology of $\mathbb{P}(E)$

(a) Let us suppose that $q_0 = \cdots = q_n = 1$. So $\widetilde{\mathbb{P}}(E) = \mathbb{P}(E)$, and $\widetilde{\xi}_E = \xi_E$ is the class of the canonical line bundle over $\mathbb{P}(E)$. It is well-known that the $H^*(X)$-module $H^*(\mathbb{P}(E))$ is free of rank $r+1$ with $\{1, \xi_E, \cdots, \xi_E^r\}$ as a basis ([BO],[J]). Moreover, the multiplicative structure of $H^*(\mathbb{P}(E))$ is determined by the unique relation ([G1],[BO],[J])

$$\xi_E^{r+1} + c_1\xi_E^r + \cdots + c_r\xi_E + c_{r+1} = 0 \qquad (c)$$

where $c_i = c_i(E)$ is the i-th Chern class of E.

(b) Now in the general case ($q_i \geq 1$), the elements $1, \widetilde{\xi}_E, \cdots, \widetilde{\xi}_E^r$ do not generate the $H^*(X)$-module $H^*(\widetilde{\mathbb{P}}(E))$. Indeed if X is one point, then $\widetilde{\mathbb{P}}(E)$ is reduced to its fiber $\widetilde{\mathbb{P}}^r$, and we know from I.7. that $1, \widetilde{\xi}_E, \cdots, \widetilde{\xi}_E^r$ do not generate $H^*(\widetilde{\mathbb{P}}^r)$.

4 The $H^*(X)$-module $H^*(\widetilde{\mathbb{P}}(E))$

(a) The result is that $H^*(\widetilde{\mathbb{P}}(E))$ is a $H^*(X)$-free module of rank $r+1$. This is done by constructing elements $w_0, \cdots, w_r$, $w_i \in H^{2i}(\widetilde{\mathbb{P}}(E))$, the restriction of which to the fiber $\widetilde{\mathbb{P}}^r$ gives the basis of $H^*(\widetilde{\mathbb{P}}^r)$ defined in Theorem I.5. This will prove that $\{w_0, \cdots, w_r\}$ is a basis for the module $H^*(\widetilde{\mathbb{P}}(E))$ by LERAY-HIRSCH theorem.

(b) We explain now how to construct the elements w_i in a particular case which is, in fact, fundamental for the general case.

So let us assume that the vector bundles $E_0, \cdots, E_n$ are line bundles and that the integers $q_0, \cdots, q_n$ are such that q_i is divisible by q_{i-1} for all $i = 1, \cdots, n$. We make a change of notation, putting $L_i = E_i$. So $E = \bigoplus_0^n L_i$, and $rk(E) = n+1$. The fiber of $\rho : \widetilde{\mathbb{P}}(E) \longrightarrow X$ is $\widetilde{\mathbb{P}}^n = \mathbb{P}_{\mathbb{C}}(q_0, \cdots, q_n)$.

(i) Since we are going to proceed by induction on n, let us first remark that there are closed immersions:

$$X = \widetilde{\mathbb{P}}(F_0) \subset \widetilde{\mathbb{P}}(F_1) \subset \cdots \subset \widetilde{\mathbb{P}}(F_n) = \widetilde{\mathbb{P}}(E) ,$$

where $F_i = L_0 \oplus \cdots \oplus L_i$ and $\widetilde{\mathbb{P}}(F_i) = \mathbb{P}(F_i; q_0, \cdots, q_i)$. These inclusions are induced by the canonical $F_i \subset F_{i+1}$. For simplification of notation, set $M_i = \widetilde{L}_{F_i}$, the canonical line bundle over $\mathbb{P}_i = \widetilde{\mathbb{P}}(F_i)$. An important fact is

$$M_i|P_{i-1} = M_{i-1}^{\otimes q_i/q_{i-1}} \quad (1 \leq i \leq n) .$$

(ii) Consider the open subset $V = \widetilde{\mathbb{P}}(E) \backslash \widetilde{\mathbb{P}}(L_n)$. This is a line bundle over $\mathbb{P}_{n-1}$ (because q_i divides q_n, $0 \leq i \leq n-1$), and

$$V = M_{n-1}^{\otimes q_n/q_{n-1}} \times p^*(L_m) \text{ , where } p : \mathbb{P}_{n-1} \longrightarrow X \text{ is the projection.}$$

In particular $H_c^*(V)$ is a free $H^*(\mathbb{P}_{n-1})$-module of rank 1, generated by the Thom class $\tau \in H_c^2(V)$ of V ([BO]).

On the other hand, there is a cohomology exact sequence

$$\cdots \to H_c^i(V) \to H^i(\widetilde{\mathbb{P}}(E)) \to H^i(\widetilde{\mathbb{P}}(L_n)) \to H_c^{i+1}(V) \to \cdots$$

The image of τ in $H^2(\widetilde{\mathbb{P}}(E))$ is $\widetilde{\xi}_E + c_1(L_m)$ $(\widetilde{\mathbb{P}}(L_m) = X)$.
(iii) Construction of the elements $w_0, \cdots, w_n \in H^*(\widetilde{\mathbb{P}}(E))$: by induction on n. Take $w_0 = 1 \in H^0(\widetilde{\mathbb{P}}(E))$. Assume that elements $w_0' = 1, w_1', \cdots, w_{n-1}'$ are defined in $H^*(\mathbb{P}_{n-1}) : w_i' \in H^{2i}(\mathbb{P}_{n-1})$, $\mathbb{P}_{n-1} = \widetilde{\mathbb{P}}(F_{n-1})$. Consider then $w_i' \cdot \tau$ in the $H^*(\mathbb{P}_{n-1})$-module $H^*(V)$ and denote by w_{i+1} its image in $H^*(\widetilde{\mathbb{P}}(E)) = H^*(\mathbb{P}_n)$.

Hence the elements $w_0, \cdots, w_n \in H^*(\widetilde{\mathbb{P}}(E))$ are well-defined. Their restriction to the fiber $\widetilde{\mathbb{P}}^n$ of $\widetilde{\mathbb{P}}(E)$ gives the basis $\xi_0 = 1, \xi_1, \cdots, \xi_n$ of $H^*(\widetilde{\mathbb{P}}^n)$ constructed in I.5. Thus, by LERAY-HIRSCH theorem, the $H^*(X)$-module $H^*(\widetilde{\mathbb{P}}(E))$ is free with $\{w_0, \cdots, w_n\}$ as a basis. Note that $w_1 = \widetilde{\xi}_E + c_1(L_n)$.

5 About the Multiplicative Structure of $H^*(\widetilde{\mathbb{P}}(E))$

The hypothesis made in 4.(b) is still valid in this paragraph. Recall that $P_i = \widetilde{\mathbb{P}}(F_i)$, $F_i = L_0 \oplus \cdots \oplus L_i$. Let $\{w^i_o, \cdots, w^i_{i-1}\}$ be the basis of the $H^*(X)$-module $H^*(P_i)$ constructed above (4.(b) iii).

(a) Consider the homomorphism $h_n : H^*_c(V) \longrightarrow H^*(\mathbb{P}_n), \mathbb{P}_n = \widetilde{\mathbb{P}}(E)$ (see 4.(b) ii). Since $h_n(\tau) = w^n_1$, and $H^*_c(V)$ is a $H^*(\mathbb{P}_{n-1})$-module, we shall write

$$a \cdot w^n_1 := h_n(a\tau) \text{ for } a \in H^*(\mathbb{P}_{n-1}) .$$

Thus, by construction of the w^j_i's, we obtain

$$\begin{aligned} w^n_0 = 1, w^n_1 &= \widetilde{\xi}_{F_n} + c_1(L_n), \\ w^n_2 = w^{n-1}_1 \cdot w^n_1 , \cdots, \ w^n_n &= w^{n-1}_{n-1} \cdot w^n_1 \\ w^{n-1}_0 = 1, w^{n-1}_1 &= \widetilde{\xi}_{F_{n-1}} + c_1(L_{n-1}), \\ w^{n-1}_2 = w^{n-2}_1 \cdot w^{n-1}_1, \ \cdots \ , w^{n-1}_{n-1} &= w^{n-2}_{n-2} \cdot w^{n-1}_1 \\ &\cdots \end{aligned}$$

So, by induction, it holds that

$$w^n_i = w^{n-i+1}_1 \cdot w^{n-i+2}_1 \cdots w^n_1 \ \ (1 \leq i \leq n) .$$

(similarly to h_n, there are homomorphisms $h_i : H^*_c(V_i) \longrightarrow H^*(\mathbb{P}_i)$, with $V_i = \widetilde{\mathbb{P}}(F_i)\backslash\widetilde{\mathbb{P}}(L_i)$ allowing to define $w^{i-1}_1 \cdot w^i_1$.) Whence

$$w^n_i = (\widetilde{\xi}_{F_s} + c_1(L_s)) \cdot (\widetilde{\xi}_{F_{s+1}} + c_1(L_{s+1})) \cdots (\widetilde{\xi}_{F_n} + c_1(L_n))$$

where $s = n - i + 1, 1 \leq i \leq n$.

Because $\widetilde{\xi}_{F_0} + c_1(L_0) = 0$ in $H^2(\widetilde{\mathbb{P}}(F_0))$ $(F_0 = L_0, \widetilde{\mathbb{P}}(F_0) = X)$, there is the following relation:

$$(\widetilde{\xi}_{F_0} + c_1(L_0)) \cdot (\widetilde{\xi}_{F_1} + c_1(L_1)) \cdots (\widetilde{\xi}_{F_n} + c_n(L_n)) = 0 \qquad (c')$$

If $q_0 = \cdots = q_n = 1$, then this relation coincides with the relation

(c) mentioned in 3.(a), which defines the Chern classes of $E = L_0 \oplus \cdots \oplus L_n$, that is:

$$(\xi_E + c_1(L_0))(\xi_E + c_1(L_1)) \cdots (\xi_E + c_1(L_n)) = 0 \ ,$$

the first member of which is the usual product in $H^*(\mathbb{P}(E))$. In fact, when $q_0 = \cdots = q_n = 1$, one has

$$w_i^n = (\xi_E + c_1(L_s))(\xi_E + c_1(L_{s+1})) \cdots (\xi_E + c_1(L_n))$$

$((s = n - i + 1)$, the second member being the usual product in $H^*(\mathbb{P}(E)))$, and -of course- these $w_0^n, w_1^n, \cdots, w_n^n$ form a basis of the $H^*(X)$-module $H^*(\mathbb{P}(E))$, but this is not the usual basis $\xi_E^0, \xi_E^1, \cdots, \xi_E^n$!

(b) So the relation (c') gives a generalization of (c). We want to write it in terms of the class $\tilde{\xi}_E$ of $\tilde{L}_E^*$ (dual of the canonical line bundle over $\tilde{\mathbb{P}}(E)$), using the usual product in $H^*(\tilde{\mathbb{P}}(E))$. This leads to the following relation in $H^*(\tilde{\mathbb{P}}(E))$

$$(\tilde{\xi}_E + c_1(L_0^{\otimes t_0}))(\tilde{\xi}_E + c_1(L_1^{\otimes t_1})) \cdots (\tilde{\xi}_E + c_1(L_n^{\otimes t_n})) = 0 \qquad (c'')$$

with $t_i = q_n/q_i$.

More generally we have

$$(\tilde{\xi}_E + c_1(L_s^{\otimes t_s}))(\tilde{\xi}_E + c_1(L_{s+1}^{\otimes t_{s+1}})) \cdots (\tilde{\xi}_E + c_1(L_n^{\otimes t_n})) = t_s t_{s+1} \cdots t_n w_i^n$$

where $s = n - i + 1, 1 \leq i \leq n$.

6 The Classes $\tilde{c}_i(E)$

We continue to suppose that the vector bundles $E_i = l_i$ are of rank 1, but no hypothesis is made about the integers $q_0, \cdots, q_n$. Let $l = \text{lcm}\{q_0, \cdots, q_n\}$. Then the relation (c'') is a particular case of the following

$$(\tilde{\xi}_E + c_1(L_0^{\otimes t_0}))(\tilde{\xi}_E + c_1(L_1^{\otimes t_1})) \cdots (\tilde{\xi}_E + c_1(L_n^{\otimes t_n})) = 0 \qquad (\tilde{c})$$

where $t_i = l/q_i$. It is this relation we consider as a relevant generalization of (c) in the case of $\widetilde{\mathbb{P}}(E)$. This will be justified in [A1]. Let $\tilde{c}_i(E)$ be the coefficient of $\widetilde{\xi}_E^{n-i+1}$ in the left member of ($\tilde{c}$), and $\tilde{c}(E) = \tilde{c}_0(E) + \cdots + \tilde{c}_n(E)$. Then

$$\tilde{c}(E) = (\psi^{l/q_0}c(L_0)) \cdots (\psi^{l/q_n}c(L_n)) ,$$

where ψ^k is the k-th Adams operation and $c(F)$ is the total Chern class of F.

Now, for general vector bundles E_i ($E = E_0 \oplus \cdots \oplus E_n$), we put

$$\tilde{c}(E) := (\psi^{l/q_0}c(E_0)) \cdots (\psi^{l/q_n}c(E_n)) ,$$

which defines classes $\tilde{c}_i(E) \in H^{2i}(X)$ $(0 \leq i \leq r = rk(E))$ such that $\tilde{c}(E) = \sum_0^n \tilde{c}_i(E)$.

These classes verify the same operations as the $c_i(E)$ except the additivity one. If $(E' = \bigoplus_0^{n'} E'_i; q'_0, \cdots, q'_{n'})$ and $(E'' = \bigoplus_0^{n''} E''_i; q''_0, \cdots, q''_{n''})$ are two examples of our datum $(E = \bigoplus_0^n E_i; q_0, \cdots, q_n)$, then it holds that

$$\tilde{c}(E' \oplus E'') = (\psi^{m/l'}\tilde{c}(E'))(\psi^{m/l''}\tilde{c}(E'')) ,$$

where $l' = \text{lcm}\{q'_0, \cdots, q'_{n'}\}, l'' = \text{lcm}\{q''_0, \cdots, q''_{n''}\}, m = \text{lcm}\{l', l''\}$, and $E' \oplus E'' = E'_0 \oplus \cdots \oplus E'_{n'} \oplus E''_0 \oplus \cdots \oplus E''_{n''}$ is weighted by the integers $q'_0, \cdots, q'_n, q''_0, \cdots, q''_{n''}$.

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