ON DEGENERATIONS OF FIBRE SPACES OF CURVES OF GENUS ≥ 2

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Abstract. In this note, we show that for surfaces admitting suitable fibrations, any given degeneration $\mathcal{X}/\Delta$ is bimeromorphic to a fiber space over Δ and we apply this result to the study of the degenerate fiber.

This note is concerned with the problem of studying the degenerations of fibered surfaces via the degenerations of the base curve and the fibers. We consider a surface of general type X admitting a fibration $X \to S$ with base genus ≥ 2 and show that for a weakly projective degeneration $\Pi : \mathcal{X} \to \Delta$ of such a surface, satisfying a mild condition on monodromy, the components of the singular fiber are of the following types:
(i) a rational surface, (ii) a ruled surface, or (iii) a surface fibered over a curve obtained from a degeneration of the base curve S.

Throughout the paper we work over $\mathbb{C}$, and adapt the following notation :
$\Delta = \{z \in \mathbb{C} : |z| < 1\}$.
$\Delta^* = \Delta - 0$.
X is a compact surface admitting a fibration $X \to S$ with general fiber F and genera $g(S), g(F) \geq 2$.
$\chi(?)$ denotes the holomorphic Euler characteristic of ?.
$K_?$ is the canonical class of ?.
$p_g(?), q(?)$ denote the geometric genus and the irregularity of ?, respectively.
$\mathcal{X}$ is a smooth threefold, $\Pi : \mathcal{X} \to \Delta$ is a flat proper holomorphic map with $\mathcal{X}_{t_0} = X$ for some $t_0 \in \Delta^*$ and the only singular fiber $\mathcal{X}_0$ is a divisor with normal crossings and smooth components.

We further assume that (i) each fiber $\mathcal{X}_t, t \in \Delta^*$ admits a fibration $\mathcal{X}_t \to S_t$ of the same type as the fibration on X and (ii) the monodromy action on $H^1(\mathcal{X}_t, \mathbb{C})$ leaves the image of $H^1(S_t, \mathbb{C})$ invariant (cf. Lemma 3 and the discussion preceding Lemma 3, for some examples of degenerations satisfying these hypothesis). We recall that under these assumptions we have the following result ([4], Proposition 1 and Proposition 3).

Theorem 1. *Possibly after restricting to a smaller disk Δ' around $0 \in \Delta$, we can find a degeneration $\Pi' : \mathcal{X}' \to \Delta'$ and a relative curve $p : \mathcal{Y} \to \Delta'$ such that :*
(i) we have a bimeromorphic map $\mathcal{X}'/\Delta' \to \mathcal{X}/\Delta'$ which is an isomorphism over Δ'^*,
(ii) Π' factors through p, $\mathcal{X}^{'} \to \mathcal{Y}$ is surjective and
(iii) for each $t \in \Delta^{'*}$, $\mathcal{X}_t^{'} \to \mathcal{Y}_t$ is the fibration $\mathcal{X}_t \to S_t$.

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Theorem 1 reduces the problem of understanding the structure of the given degeneration to the study of the degenerations of the curves S_t and of the fibres of $X \to S$. In this direction, without any projectivity assumptions on $\mathcal{X}/\Delta$, we have

Lemma 1. *If a component X_i of $\mathcal{X}_0^{'}$ maps onto a component Y_i of $\mathcal{Y}_0$, then it is algebraic.*

Proof : If $X_i \to Y_i$ is a fibration of fiber genus 0 or ≥ 2 then clearly X_i is algebraic. So we assume that $X_i \to Y_i$ is an elliptic fibration. Since a surface with algebraic dimension $= 0$ has only a finite number of curves, X_i has algebraic dimension $= 1$ or 2. As the fibers of $\mathcal{X}_0^{'} \to \mathcal{Y}_0$ are curves of genus ≥ 2 (except at a finite number of points where the map is not flat), we see that on X_i lie other curves obtained from intersection with the components of $\mathcal{X}_0^{'}$ containing the rest of the fibers of $\mathcal{X}_0^{'} \to \mathcal{Y}_0$ over Y_i. Clearly, such curves will not lie in the fibers of the elliptic fibration on X_i and this is impossible unless X_i has algebraic dimension 2. This proves the lemma. □

Lemma 2. *Let X_j be a component of $\mathcal{X}_0^{'}$ mapped to a point $p \in \mathcal{Y}_0$. If X_j intersects some component X_i as in Lemma 1 along a smooth curve C with no triple points, then X_j is algebraic.*

Proof : Let $C_i(resp.C_j)$ denote the curve C on $X_i(resp.X_j)$.
If C_i is the fiber of X_i over p, then $C_i^2 = 0$. Therefore, as there are no triple points on C we have $C_j^2 = -C_i^2 = 0$. Moreover $g(C) = g(F) \geq 2$ and using the adjunction formula on X_j, we get $K_j.C_j + C_j^2 = 2(genus(C) - 1) \geq 2$. Hence $(K_j + nC_j)^2 > 0$ for large enough n and therefore X_j is algebraic.
On the other hand, if C_i is a component of the fiber over p, then $C_i^2 < 0$ and the equality $C_j^2 = -C_i^2$ gives $C_j^2 > 0$, again proving the algebraicity of X_j. □

These two lemmata clearly fall short of proving the algebraicity of all components of the singular fiber (cf. [5], conjecture on p. 83). However, combining Lemma 1 with the flatification technique of ([2]) we get

Theorem 2. *$\mathcal{X}/\Delta^{'}$ is bimeromorphic to a degeneration $\mathcal{X}^{''}/\Delta^{'}$ in which all components of the singular fiber $\mathcal{X}_0^{''}$ are algebraic.*

Proof : To prove this result we first remove those components of $\mathcal{X}_0^{'}$ where the map $\mathcal{X}^{'} \to \mathcal{Y}$ fails to be flat. For this purpose, we will apply flatification as described in ([2]). More precisely, we blow up points $p_1, .., p_k \in \mathcal{Y}_0$ over which our map is not flat, to get a new relative curve $\mathcal{Y}^{'}/\Delta^{'}$ and then in the complex space $\mathcal{X}^* = \mathcal{X}^{'} \times_{\mathcal{Y}} \mathcal{Y}^{'}$ we take the smallest closed analytic subspace $\mathcal{X}^{**}$ containing $\mathcal{X}^* - \cup \Pi^{*-1}(p_j)$ where Π^* is the composite map $\mathcal{X}^* \to \mathcal{Y}^{'} \to \mathcal{Y}$. Then $\mathcal{X}^{**} \to \mathcal{Y}^{'}$ is flat. Finally, resolving the singularities of $\mathcal{X}^{**}$ and of the components of the singular fiber, we get the required degeneration $\mathcal{X}^{''}/\Delta^{'}$. □

Corollary 1. *If $\mathcal{X}/\Delta$ is weakly projective, then each component of $\mathcal{X}_0$ is either a fibration over a curve of genus $\leq g(S)$ with fiber genus $\leq g(F)$ or a ruled or rational surface.*

Proof : Since $\mathcal{X}$ is weakly projective, so is $\mathcal{X}''$ of Theorem 2 and we apply ([5], Corollary 3.1.4). □

Next we address to the question of when the hypothesis of Theorem 1 are satisfied for a degeneration $\mathcal{X} \to \Delta$. As to the first condition we have
a) if $X \to S$ is a smooth fibration with both fiber genus and base genus ≥ 2, then any deformation of X admits a fibration of the same type ([3], Lemma 7.1), and
b) if the fibration is a consequence of a relation among some deformation invariants, then trivially the first condition of the hypothesis holds. One notable example of this case is degenerations of minimal surfaces with $K^2 < 3\chi, q \geq 2$. With this inequality satisfied, the given surface admits a fibration of fiber genus 2 or 3, the base curve being the image of the albanese map ([1], Theorem 2.6).
The condition on monodromy is trivially satisfied if $h^1(|\Gamma|) = 0$ where Γ is the dual graph of the singular fiber. For more general degenerations we have

Lemma 3. *In the following cases the hypothesis on monodromy is satisfied :*
(a) $X \to S$ *is a smooth fibration with* $g(S), g(F) \geq 2$,
(b) X *is minimal and* $K^2 < 3\chi(X), q(X) \geq 2$.

Proof : (a) By ([3], Lemma 7.1), we have a deformation $\Phi : \mathcal{S} \to \Delta^*$ of S, varying continuously with $t \in \Delta^*$, such that $\Pi|_{\Delta^*}$ factors through Φ. Therefore, we have an exact sequence $0 \to R^1\Phi_*(\mathbb{C}) \to R^1\Pi_*(\mathbb{C})$. Hence, the correspondence between representations of $\pi_1(\Delta^*)$ and flat vector bundles on Δ^* shows that, for each $t \in \Delta^*$, $H^1(S_t, \mathbb{C})$ is an invariant subspace of $H^1(\mathcal{X}_t, \mathbb{C})$ under the monodromy action.

(b) In this case the fibration Ψ_t being the albanese fibration we have $H^1(\mathcal{X}_t, \mathbb{C}) = H^1(S_t, \mathbb{C})$ and the conclusion follows trivially. □

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