Arf Rings and Characters

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Dedicated to Ordinarius Professor Cahit Arf
on the occasion of his eighty fifth birthday

1 Introduction

In 1949 an article by Cahit Arf [1] appears in the Proceedings of the London Mathematical Society. In this article Arf solves the classification problem of singular curve branches based upon their multiplicity sequence. However, the geometric nature of the problem is so hidden behind the algebraic ideas and subsequent constructions that immediately following Arf's article an article by Du Val [6] appears, beginning with the words: "Cahit Arf's results being severely algebraic in form,...". There Du Val provides the general reader with the necessary geometric ideas that lie behind the scene. This is most appropriate since it was Du Val who had formulated the final form of the problem in his Istanbul article [5] to which Arf refers quite warmly in his Proceedings article.

In this exposition I will attempt to describe both the problem and the solution, using as few machinery as possible, and mostly in today's terminology. The problem and its solution form one of those rare occasions where the solution supplied by an article answers the question of another article, and does so without altering the question to suit the answer! I will describe this solution and then discuss some future prospects. However, I must add that neither Arf nor Du Val can be held responsible for the yet unsubstantiated optimism that surrounds the ideas I will express in the concluding remarks of this article. I rely on the dexterity of my students to acquit me in history for any hopes expressed in the final section.

In 1985, fresh out of the graduate school, I was hired by Erdoğan Şuhubi as a research assistant to TÜBİTAK's Gebze Research Center. The person to whom I would assist and collaborate with was none other than Cahit Arf himself. In this article while trying to convey the joy and excitement that dominated our discussions on this topic I may inadvertently give away some trade secrets that I learned from him, for which I apologize from him beforehand. It is a privilege for me to dedicate this work, albeit humble, to Professor Cahit Arf on the occasion of his eighty fifth birthday, with gratitude and respect.

2 The Setup and the Problem

2.1 Heuristic Arguments

The main problem dominating the area is to understand the behaviour of a curve at its singularity. Before we attempt any definition however, we must agree to choose and fix an underlying field to work with. Let us call our base field $\mathbf{k}$. For drawing pictures and extracting intuition $\mathbf{k}=\mathbb{R}$ is appropriate but for most geometric applications $\mathbf{k}=\mathbb{C}$ is used. Algebraic geometers generally chose $\mathbf{k}$ to be an algebraically closed field of any characteristic. Arf's arguments however will work for any field $\mathbf{k}$ of any characteristic.

2.2 The First Example: The Node

Let us begin with an example. Consider the curve C defined by the equation $y^2 = x^2(x+1)$ in the affine plane $\mathbb{A}^2_{\mathbf{k}}$. The curve C is the nodal curve which has a singularity at the origin where the curve intersects itself. The usual way of 'correcting' this singularity is by changing the space in which the curve lies and hoping that the curve will behave better given a more suitable environment. For this we blow up $\mathbb{A}^2$ at the origin and consider the monoidal transform of the curve in the blow up. This involves first replacing $\mathbb{A}^2$ by the new space

$$B_2 = \{((x,y), [u_1 : u_2]) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid u_1 y = u_2 x \} \tag{2.1}$$

which is easily seen to be smooth. If $\pi : B_2 \to \mathbb{A}^2$ is the projection on the $\mathbb{A}^2$ component then $\pi^{-1}(0,0)$ is isomorphic to $\mathbb{P}^1$. It is denoted by E and is called the exceptional divisor. Note that $\pi^{-1}(\mathbb{A}^2 - (0,0))$ is isomorphic to $B_2 - E$. In particular $\pi^{-1}(C - (0,0))$ is isomorphic to $C - (0,0)$. So we have carried the smooth part $C - (0,0)$ of the singular curve C to a new space B_2. Now we look for a substitute for the missing point. A natural way of doing this is by taking

the closure of $\pi^{-1}(C - (0,0))$ in B_2, which we denote by $\tilde{C}$. The question now is whether $\tilde{C}$ is smooth or not. For this we consider local coordinates on B_2 and examine the equation of $\tilde{C}$ in these coordinates. Let U_i be the subset of B_2 consisting of the points with $u_i \neq 0$, $i = 1, 2$. Note that $\{U_1, U_2\}$ is an open cover for B_2. Define the local coordinates of B_2 as

$$X = x, \quad Y = u_2/u_1 \quad \text{in } U_1, \tag{2.2}$$
$$X = y, \quad Y = u_1/u_2 \quad \text{in } U_2. \tag{2.3}$$

With this notation the exceptional divisor E intersects U_i along the line $X = 0$, for $i = 1, 2$. The equation of $\tilde{C}$ becomes

$$Y^2 = X + 1 \qquad \text{in } U_1 \tag{2.4}$$
$$1 = XY^3 + Y^2 \qquad \text{in } U_2. \tag{2.5}$$

We see that $\tilde{C}$ is smooth in both of these coordinate neighbourhoods. Moreover $\tilde{C}$ intersects E at the points $(0, \pm 1)$ in both charts. The numbers in the y-component correspond to the slopes of the tangent lines to C at the origin in $\mathbb{A}^2$.

2.3 The Second Example: The Cusp

Next consider another example; Define C' to be the cusp in $\mathbb{A}^2$ given by the equation $y^2 = x^5$. Denote by $\tilde{C}'$ the closure of $\pi^{-1}(C' - (0,0))$ in B_2. The equation of $\tilde{C}'$ becomes

$$Y^2 = X^3 \qquad \text{in } U_1 \tag{2.6}$$
$$1 = X^3Y^5 \qquad \text{in } U_2. \tag{2.7}$$

We see that $\tilde{C}'$ intersects E at the point $(0,0)$ in U_1 and is singular there whereas it is smooth in U_2 and does not intersect E there. Judging from the way the equation of C' is transformed under the blow up operation we conclude that if we apply another blow up operation to $\tilde{C}'$ in U_1 at $(0,0)$ then the curve will be transformed to a smooth curve of the form $Y^2 = X$. The tangent line to the curve C' at the origin is horizontal and this is reflected in the Y component of the point where $\tilde{C}'$ intersects E in U_1.

2.4 The General Case

The crucial information coded in the singularity seems to surface at the intersection points of the transformed curve with the exceptional divisor. If we could

work with 'one piece of information' at a time, then after each blow up there would be only one intersection with the exceptional curve and we would continue our analysis from there on. For this purpose we restrict our attention to such pieces of information at each singular point on the curve. When we later make this concept precise we will call it a branch of the curve.

One significant information about the singular point is the multiplicity of that point. To find the multiplicity of a point we first count the number of points a general line intersects the curve. This number is also known as the degree of the curve. Then we consider a general line passing through the singular point and count at how many other points it intersects the curve. We subtract this number from the degree of the curve and call it the multiplicity of the singular point. This is reasonable since this difference must count the contribution of that singular point. Algebraically speaking, a plane curve is given by a polynomial of degree n and the number of intersection points of this curve with a line corresponds to the number of roots of this polynomial after a linear substitution is made. The number of roots is equal to the degree of the polynomial when **k** is algebraically closed. The situation is similar in n-space.

We can thus associate to each singular point its multiplicity. The multiplicity of a smooth point is 1 by the above definition. Assume that p_0 is a singular point of a curve and that the blow up of the curve at this point intersects the exceptional divisor at only one point and further more assume that the same is true for the subsequent transforms. This property ad hocly describes a singular curve branch. We then obtain a sequence $\{(p_i, m_i)\}$, where p_i is obtained from p_{i-1} by blowing up and m_i is the multiplicity of p_i. For ease of notation we can only consider the sequence $\{m_i\}$ which is called the multiplicity sequence of the branch. We now agree to call two singular branches equivalent if their multiplicity sequences are the same. The problem is then to classify all singular branches up to this equivalence class.

2.5   Technical Formulation

We observed in the previous sections that the nodal curve C had two parts to its singularity at the origin whereas the cuspidal curve C' had only one. How can we recognize this phenomena by looking at their equations? Clearly we wish the equation of C to split up as the product of two parts and the equation of C' remain irreducible. The expression $y^2 - x^2(x+1)$ is irreducible in the ring $\mathbf{k}[x, y]$. We may say that this ring is unnecessarily small since it corresponds to global polynomial functions on $\mathbb{A}^2$, whereas we are interested only in what happens at the origin. Therefore we can look at $k[x, y]_{(x,y)}$, the localization of $k[x, y]$ at its maximal ideal (x, y). This ring represents the regular functions at

the origin and should fit to our geometric purpose of focusing our attention to the origin. However $y^2 - x^2(x+1)$ is still irreducible in this ring. This hints to us that we are probably not working in the right rings. Each irreducible component of $y^2 - x^2(x+1)$ should be of the form $y = \pm x\sqrt{x+1}$. But $\sqrt{x+1}$ is not an element of the rings $\mathbf{k}[x,y]$ and $k[x,y]_{(x,y)}$. Therefore we must find a ring in which $\sqrt{x+1}$ exists. Observe however that $x+1$ is the square of 1 when computed modulo the maximal ideal corresponding to the origin. This suggests that we should look at the Henselization of the local ring $k[x,y]_{(x,y)}$. (see [11, 7] for a discussion of Henselization.) On the other hand the completion of $k[x,y]_{(x,y)}$ with respect to its maximal ideal always satisfies Hensel's lemma and it can be used at this stage. In fact in the formal power series ring $\mathbf{k}[[x,y]]$ we can write $\sqrt{x+1} = \pm(1 + x/2 - x^2/8 + \cdots)$. Hence the equation $y^2 - x^2(x+1) = 0$ splits up as $(y - x(1 + x/2 - x^2/8 + \cdots))(y + x(1 + x/2 - x^2/8 + \cdots))$.

The irreducibility of the expression $y^2 - x^2(x+1)$ corresponds to the fact that $\mathbf{k}[x,y]/(y^2 - x^2(x+1))$, the ring of polynomial functions on $\tilde{C}$, is an integral domain. We are interested in what happens at the origin so we localize with respect to the maximal ideal corresponding to the origin. We can in fact first localize and then consider the quotient to obtain the ring $k[x,y]_{(x,y)}/(y^2 - x^2(x+1))$. This is an integral domain. Completing this ring with respect to its maximal ideal we obtain a ring with zero divisors since $y^2 - x^2(x+1)$ which corresponds to zero can be split up as in the above discussion. Note however that the equation $y^2 - x^5$ continues to stay irreducible even after completing the relevant ring. In fact x is never a square in the ring $R[x]$ where R is a commutative ring with unity. It can only be a square if R is a suitably chosen noncommutative ring. In our case the coefficient rings are always fields so x will never split. (for a discussion of localizations, quotients and completions see [14, 2].)

We can now give a technical definition for a curve branch. Consider the prime ideal describing an irreducible curve in n space with a singularity at the origin. The ideal it generates inside the formal power series with n indeterminates may split up into components. Each such component is a branch of the curve passing through the origin. For a geometric description see [13].

2.6   Du Val's Formulation

In a much neglected article [5] Du Val summarizes the stage for the classification problem of singular curve branches and formulates the question whose answer he claims will lead to a complete understanding of the situation. It is left to the reader to check that the description of the problem in this section agrees with the one given in the previous sections.

Define a curve branch C in n-space by the following formal parameterization;

$$
\begin{aligned}
x_1 &= \phi_1(t) \\
x_2 &= \phi_2(t) \\
\vdots \quad &\vdots \quad \vdots \\
x_n &= \phi_n(t)
\end{aligned}
\tag{2.8}
$$

where each $\phi_i(t)$ is a formal power series in t with coefficients from the field $\mathbf{k}$. We want the branch to pass through the origin so we impose the condition that the constant term of each $\phi_i(t)$ is zero. Assume that the order of $\phi_1(t)$ is lowest among the others. We want to blow up the branch at the origin and write the parameterization of the transformed branch in the coordinate chart where it intersects the exceptional divisor. The blow up of $\mathbb{A}^n$ can be described as

$$
B_n = \{((x_1, ..., x_n), [a_1 : \cdots : a_n]) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid x_i a_j = x_j a_i, \ 1 \leq i, j \leq n \}. \tag{2.9}
$$

In U_1 the local coordinates can be written as

$$
X_1 = x_1, \ X_2 := a_2/a_1, ..., X_n = a_n/a_1. \tag{2.10}
$$

Since $\phi_1(t)$ was chosen to have the smallest order, U_1 is the chart in which the transformed branch intersects the exceptional divisor. The parameterization of the transformed branch is now given by

$$
\begin{aligned}
X_1 &= \phi_1(t) \\
X_2 &= \phi_2(t)/\phi_1(t) \\
\vdots \quad &\vdots \quad \vdots \\
X_n &= \phi_n(t)/\phi_1(t).
\end{aligned}
$$

Observe that each X_i is again expressed as a formal power series in t. The multiplicity of a branch given by such a representation is equal to the order of the lowest order series appearing in the representation.

If the branch C has its singularity p_1 located at the origin then its blow up intersects the exceptional divisor E_1 of the first blow up at a point p_2. Semple in [12] defines p_2 to be proximate to p_1. The second blow up is centered at p_2 and intersects the exceptional divisor E_2 of the second blow up at p_3. The transform of E_1 also intersects E_2 at p_{12}. If $p_3 = p_{12}$ then we say that p_3 is proximate to p_1 and p_2. Otherwise it is proximate only to p_2. In general if a certain p_{i+j}, with $i, j > 0$ lies on E_i or on any transform of E_i under the subsequent blow ups then we say that p_{i+j} is proximate to p_i. Nowadays we use the term "infinitely close" instead of "proximate". (see [12, 8]). The sum of the multiplicities of the points p_1, p_2,...,p_m is called the m-th multiplicity sum.

Du Val in [5] a point p_i as a leading point if the number of points to which it is proximate is less than the number of points to which p_{i+1} is proximate. After this bombardment of definitions comes the most relevant definition: the multiplicity sum corresponding to a leading point is called a character of the curve. In other words if p_i is proximate to less points than p_{i+1} then the sum of the multiplicities of the points p_1,..., p_i is called a character of the branch. The set of all the characters of the curve was later called the Arf characters. To the best of my knowledge the first person who first observed the significance of these numbers and called them "characters" was Du Val in [5] and Arf was the first person who could explicitly calculate them, in [1].

Du Val then defines an algorithm, which he calls the modified Jacobian algorithm, to calculate the multiplicity sequence when these characters are known. (The Jacobian algorithm, and also the modified Jacobian algorithm for that matter, calculates the greatest common divisor of a given set of positive integers.) Finally at the end of [5] he opens a section with the formidable title of "Outstanding Questions" and lists some natural questions related to characters.

Question 2.6.1 (Du Val) *How do you find the characters of a branch if only the local parameterization with formal power series is available?*
(see 3.4.1 for a complete answer).

Question 2.6.2 (Du Val) *Given a set of positive integers how do you know that they are the characters of some branch?*
(see 3.4.2 for a complete answer).

Question 2.6.3 (Du Val) *Can you find the smallest dimensional space into which the curve branch can be projected without changing its multiplicity sequence?*
(see 3.4.3 for a complete answer).

Arf recalls that he objected to the amount of geometric consideration that was clouding the problem, when Du Val first gave a talk on this subject at Istanbul University. It must have been 1945. He claimed that there was a very algebraic pattern in the problem which could be solved if one could forget the great geometrical significance of the problem. Naturally Du Val asked him to work on this. The next day Arf was homebound with a severe cold so he decided he might as well think about this problem. Next week when he returned to work he had in his pocket, scribbled as usual on small pieces of paper, his own ticket to immortality...!

3 The Solution

In this section we will describe the tools that are developed by Arf to solve the above problem. In the course of these descriptions it may seem to the reader that we have strayed away from the problem. But despite mounting evidence against we will be doing geometry and all this will be justified at the end when we describe how these pieces fall in to complete the jigsaw.

3.1 Generalities and Some Notation

We will be working in the formal power series ring $\mathbf{k}[[t]]$ of a single indeterminate t. If H is a subring of $\mathbf{k}[[t]]$ then we define

$$
\begin{aligned}
W(H) &= \{ord\alpha \mid \alpha \in H \} & (3.11)\\
&= \{i_0 = 0 < i_1 < \cdots < i_r < \ldots \} & (3.12)
\end{aligned}
$$

The integers $i_0, i_1, \ldots$ form a semigroup of the additive group of nonnegative integers $\mathbb{N}$. We assume that H is always so chosen that the semigroup $W(H)$ contains all integers large enough. In other words if ν_l denotes the greatest common divisor of the integers $i_1, i_2, \ldots, i_l$, then for ρ large enough we want $\nu_\rho = 1$. This is not a serious restriction since if H does not satisfy this condition then H can be transformed by an automorphism of $\mathbf{k}[[t]]$ into a subring H' which satisfies this condition. Arf does not carry out this transformation but chooses and fixes a appropriate $T \in \mathbf{k}[[t]]$ whose order is $\nu = gcdW(H)$, and assumes throughout that his ring H can be considered as a subring of the power series ring in the variable T if necessary, see [1, p 258, Remarque].

For each i_r in $W(H)$ let S_{i_r} be an element of H with $\mathrm{ord}S_{i_r} = i_r$. We define an ideal I_h by

$$
I_h = \{\alpha \in H \mid ord\alpha \geq h \}. \qquad (3.13)
$$

It can be shown that the inverse of any element in H of order zero is again an element of H. With this in mind we define the **set** I_h/S_h as

$$
I_h/S_h = \{\alpha S_h^{-1} \in H \mid \alpha \in I_h \}. \qquad (3.14)
$$

This set consists of certain elements of H closed under addition since I_h is an ideal. But the product of any two elements from this set need not be in the set. We want to consider the smallest subring of H containing the set I_h/S_h. It turns out that this ring is independent of the particular element S_h we have

chosen. So we define the **ring**

$$[I_h] \quad = \quad \text{the smallest subring of } H \text{ containing } I_h/S_h. \qquad (3.15)$$

Similarly for a semigroup $G = \{i_0 = 0 < i_1 < i_2 < \cdots\}$ of nonnegative integers and for $h \in G$ define

$$
\begin{aligned}
G_h &= \{\alpha \in G \mid \alpha \geq h\,\} & (3.16)\\
G_h - h &= \{(\alpha - h) \in G \mid \alpha \geq h\,\} & (3.17)\\
[G_h] &= \text{the semigroup of nonnegative integers} \\
&\quad \text{generated by the set } G_h - h. & (3.18)
\end{aligned}
$$

3.2 Arf Rings and Arf Semigroups

It is clear now that only for special rings H the set I_h/S_h will already be a ring. We single out such rings and give them a name:

Definition 3.2.1 (Arf Ring) [1, p 260]
A subring H of the formal power series ring $k[[t]]$ is called an Arf ring if the set I_h/S_h is a ring for every nonzero $S_h \in H$. (i.e. $I_h/S_h = [I_h]$ for every $S_H \in H$. See the equations 3.14 and 3.15.)

Similarly we select out special semigroups:

Definition 3.2.2 (Arf Semigroup) [1, p 260]
A semigroup G of nonnegative integers is called an Arf semigroup if the set $G_h - h$ is a semigroup for every $h \in G$. (i.e. $G_h - h = [G_h]$ for every $h \in G$. See the equations 3.17 and 3.18.) 3.15)

What if a ring H does not satisfy this condition? We then associate to it a ring which does:

Definition 3.2.3 (Arf Closure of a Ring) [1, p 263]
If H is a subring of $\mathbf{k}[[t]]$ then we define the Arf closure $^*H$ of H to be the smallest Arf ring in $\mathbf{k}[[t]]$ containing H.

We similarly define Arf closure for semigroups:

Definition 3.2.4 (Arf Closure of a Semigroup) [1, p 263]
If G is a subsemigroup of the nonnegative integers $\mathbb{N} = \{0, 1, 2, ...\}$ then we define the Arf closure $^*G$ of G to be the smallest Arf semigroup in $\mathbb{N}$ containing G.

It remains to check that these definitions are not void. The ring $\mathbf{k}[[t]]$ itself is obviously an Arf ring. So for any subring H the collection of Arf rings in $\mathbf{k}[[t]]$ containing H is not empty and by Zorn's lemma must have a smallest element which we call $^*H$. Here ordering is done with respect to inclusion. Similarly the semigroup $\mathbb{N}$ is an Arf semigroup and thus the definition of Arf closure for semigroups is not void.

3.3 Arf Characters

In the previous section we saw that we are building parallel constructions in algebra and arithmetic but it was not clear from their definitions that they would interact in a meaningful way. We are now ready to observe a crucial interaction. If H is a subring of the power series ring, as described in section 3.1, first take its Arf closure $^*H$ and then look at $W(^*H)$, the semigroup of orders of the Arf closure. There is a smallest semigroup g_χ in $\mathbb{N}$ whose Arf closure is equal to $W(^*H)$. This semigroup g_χ has a minimal generating set $\chi_1, ...\chi_n$ which generates it over $\mathbb{N}$. These are the Arf characters of H:

Definition 3.3.1 (Arf Characters) [1, p 265]
If H is a subring of $\mathbf{k}[[t]]$ as described in 3.1, then the characteristic semigroup of H is defined to be the smallest semigroup g_χ in $\mathbb{N}$ with $^*g_\chi = W(^*H)$. The semigroup g_χ can be generated over $\mathbb{N}$ with a minimal set of positive integers $\chi_1, ..., \chi_n$ which are defined to be the characters of H.

As is mentioned in section 2.6 the concept of characters seems to have originated with Du Val's article [5]. However the following ideas appear for the first time in Arf's article [1] for the explicit purpose of solving the problem raised in question 2.6.3.

Definition 3.3.2 (Bases, Base Characters, Dimension) [1, pp 271-274]
If H is an Arf ring let X_1 be an element in H of smallest positive order. $X_1, ..., X_{n-1}$ having been chosen let X_n be an element of smallest order in H not included in the Arf closure of the ring $\mathbf{k}[X_1, ..., X_{n-1}]$. (The ring $\mathbf{k}[X_1, ..., X_r]$ is defined as consisting of the elements of the form $\sum \alpha_{j_1 j_2 \cdot j_r} X_1^{j_1} X_2^{j_2} \cdots X_n^{j_r}$ where $\alpha_{j_1 j_2 \cdots j_r} \in \mathbf{k}$ and the summation is taken over all $(j_1, j_2, ..., j_r) \in \mathbb{N}^r$.) Since $W(H)$ is finitely generated over $\mathbb{N}$ this process terminates and we obtain

a finite collection of elements $\{X_1, ..., X_m\}$. Such a collection is called a base of H. If we denote their orders by $\chi_i = ord(X_i)$ for $i = 1, ..., m$, then the numbers $\chi_1, ..., \chi_m$ are called the base characters of H. The number m is called the dimension of H.

These definitions are backed up with concrete constructions. If G is a semigroup of $\mathbb{N}$ then a method of constructing its Arf closure is described in [1, p 263]. If H is a subring of $\mathbf{k}[[t]]$ the a method of constructing its Arf closure is given in [1, p 267]. With these methods the characters can be calculated in a finite number of steps. But not content with this Arf gives a direct method of computing the characters of any Arf semigroup, [1, p 277].

3.4   Solving the Problems

The setup being as in section 2.6 we now explain the procedure for answering the questions 2.6.1, 2.6.2 and 2.6.3. At our disposal we only have some elements $\phi_1(t), ..., \phi_n(t)$ of $\mathbf{k}[[t]]$ coming from the parameterization of the branch C as in the equation 2.8. We now answer the question 2.6.1.

Answer 3.4.1 *Denote by H the ring $\mathbf{k}[\phi_1(t), ..., \phi_n(t)]$ generated by the $\phi_i(t)$'s in the formal power series ring $k[[t]]$. (See the definition 3.3.2 for a description of the ring $\mathbf{k}[\phi_1(t), ..., \phi_n(t)]$.) First construct its Arf closure $^*H$ and then construct the smallest semigroup g_χ whose Arf closure is $W(^*H)$. The minimal generators of g_χ are the characters of the given branch C. Or alternatively use the method described in [1, p 277] to find the characters directly from $W(^*H)$.*

The validity of this answer is proved in [1, p 266, Théorème 3] where Arf shows that Du Val's Jacobian algorithm applied to these characters gives the sought for multiplicity sequence.

Now we come to the question 2.6.2 of knowing whether a given set of positive integers $0 < \gamma_1 < \cdots < \gamma_l$ are characters of an actual branch.

Answer 3.4.2 *If G denotes the semigroup of $\mathbb{N}$ generated by $0 < \gamma_1 < \cdots < \gamma_l$, then the characters of $^*G$ form a subset of $\{\gamma_1, ..., \gamma_l\}$, [1, p 256, Théorème 2]. If any element from the semigroup $^*G$ is added to this set of characters then the resulting set of integers will give the same multiplicity sequence when Du Val's modified Jacobian algorithm [5, p 108] is applied to them, [1, p 266, Théorème 3]. Once $^*G$ is known one can construct elements $\phi_1(t), ..., \phi_r(t)$ of $\mathbf{k}[[t]]$ such that the ring $^*H = \mathbf{k}[\phi_1(t), ..., \phi_l(t)]$ is an Arf ring and that $W(^*H) = ^*G$, [1, p 277 and p 282, Théorème 6].*

And we finally come to the question 2.6.3 of finding the smallest dimensional space into which the branch C given by the equations 2.8 can be projected without changing its multiplicity sequence.

Answer 3.4.3 *If H denotes the ring $\mathbf{k}[\phi_1(t), ..., \phi_n(t)]$, defined as in the definition 3.3.2, then the space of smallest dimension into which the branch C can be projected without changing its multiplicity sequence has its dimension equal to the dimension of $^*H$. (Recall the definition of the dimension of $^*H$ as given in the definition 3.3.2.)*

To prove the validity of this answer Arf defines in [1, p 273] a generating system as a set of elements $Y_1(t), ..., Y_\nu(t)$ in H such that the Arf closure of $\mathbf{k}[Y_1(t) - Y_1(0), ..., Y_\nu(t) - Y_\nu(0)]$ is equal to $^*H$. Then on [1, p 274] he shows that the smallest number of elements required in a generating system is the dimension of $^*H$. And finally in [1, p 279, Théorème 5] he shows how to construct a ring $^*H$ of a given set of characters such that $^*H$ has the smallest possible dimension.

For the reader who wants to see only a geometric argument summarizing all this we refer to the last section of Arf's article [1, pp 285-287] where he relates all this "algebraic interpretation" to the problems raised by Du Val in [5].

3.5 Silent Heros: Base Characters

In the statements of the above answers it was not necessary to quote even the existence of base characters let alone their importance. However they play a crucial role in shedding light into the whole scene besides actually answering question 2.6.3. To understand the role they play in this set up first observe that if $^*H$ is an Arf ring then $^*H_h := [I_{i_h}]$ is also an Arf ring. (See section 3.1 for the notation. In particular note that $W(^*H) = \{i_0 < i_1 < i_2 < \cdots\}$ and hence the subscript h counts the number of possible constructions up to that stage.) Arf shows that the characters of each $^*H_h$ are determined by characters of $^*H$ but the base characters of each $^*H_h$ constitute a new set of characters. And he then sets out to demonstrate how they can be constructed, [1, pp 274-275]. Thus he obtains for each $^*H$ a set of invariants: the characters of $^*H$ and the base characters of $^*H_h$ for each $h \geq 0$. Moreover in [1, p 282, Théorème 6] he shows that if l_c denotes the number of characters of $^*H$ and if l_b denotes the least possible number of base characters of $^*H$ (which he shows how to construct in [1, p 279, Théorème 5]) then for any integer n with $l_b \leq n \leq l_c$ there exists an Arf ring who has the same characters and is of dimension n. To show that all this is possible he calculates a concrete example in [1, pp 283-284] where he starts with an Arf semigroup $^*G$ and first calculates its characters. Then

he calculates all possible sets of invariants that can be associated to $^*G$, as he explained how to do in [1, pp 277-278], and finishes the example by producing an actual $^*H$ for each set of invariants having that set as its set of invariants. So a careful rereading of Arf's article shows that the algebraic structure of a branch is totally understood with the help of base characters. Despite their importance they have never been given the "Arf" adjective which they silently deserved...

4   Concluding Remarks

The most significant follow up of Arf rings came with Lipman's 1971 article [10] in American Journal of Mathematics. In fact it seems that Lipman was the first mathematician to coin the expression "Arf Rings" in the literature. He seems to have been motivated by the similarity of ideas in Arf rings and in Zariski's theory of saturations. In this article he studies the condition that Arf singled out for his rings and relates them to Zariski's ideas in analyzing the singularities.

The basic idea seems simple; if there is a singularity then the local ring there 'misses' something and the idea is to fill these 'gaps' in a controlled manner so as to understand the nature of the singularity. A similar idea in a much grand scale was utilized by Hironaka in [9] where he measures how far his local rings are from being regular. Then Bennett showed that in higher dimensions Hilbert functions can be used effectively to measure these 'gaps', see [3]. Arf's idea, being simple and fundamental surfaces every now and then in the study of singularities. Recently the Italian and Spanish mathematicians are working on ideas around Arf rings.

As already mentioned in section 2.5 the completion of the local ring at the singularity may be too large. Henselization may be enough to find a complete set of invariants but the calculations may be involved, to say the least. In higher dimensions instead of checking the deviation from regularity by Hilbert functions I believe that a direct description of the 'gaps' may be much more useful. The recent developments on Gröbner bases provides a margin of hope that this is possible. This particular speculation aims to provoke several definitions trying to pinpoint a feature in the ring as a 'gap'... On the other hand the innocent looking structure of a branch involves questions about the structure of complete rings which are for some cases answered satisfactorily by Cohen in [4], however the theory is far from being complete.

When the topic is on curves one can speculate forever! But past the basic definitions comes a land of no man... I remember what I heard years ago in a

conference on curves: "To learn modesty one must study the curves."

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