

Genus Calculations of Complete Intersections

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1 Introduction

In this article we calculate the genus of a projective complete intersection of any dimension and the genus of an affine complete intersection curve of superelliptic type. The arithmetic genus of a projective complete intersection depends on the degrees of the hypersurfaces cutting it. A formula for its genus using only the degrees of these hypersurfaces is given by Hirzebruch in [3] as an application of his Riemann-Roch theorem. Miranda has communicated us the following algorithm [5]:

One has to compute the cohomology $H^i(\mathcal{O}_X(k))$ for every complete intersection $X \subset P^n$, for every twist k , and for every cohomology group H^i . Start with $X = P^n$ itself, where all is known. Assume you know everything when Y is a complete intersection with degrees (d_1, d_2, \dots, d_r) and let X be a complete intersection with degrees $(d_1, d_2, \dots, d_r, d_{r+1})$. Hence X is the intersection of Y with a hypersurface of degree d_{r+1} . Then use the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(k - d_{r+1}) \longrightarrow \mathcal{O}_Y(k) \longrightarrow \mathcal{O}_X(k) \longrightarrow 0.$$

Knowing the cohomology on the left and the middle will tell you the cohomology on the right. Applying this recursive algorithm with X a curve gives the genus computation as $g = h^1(\mathcal{O}_X)$.

We calculate the Hilbert polynomial of a projective complete intersection by using the Koszul resolution. The arithmetic genus is then obtained from the constant term of Hilbert polynomial.

In the affine case the usual procedure for finding the genus of a curve is to find the genus of a resolution of its projective closure. This is intractable in general. We restrict our attention to the types of curves which are used in coding theory. These are superelliptic curves which we describe in equation (5). Stepanov and Özbudak has calculated the genus of some elliptic type curves in [9] using the “counting the differentials” method (for an account of this, see [7]). Stepanov also calculates the genus of an important class of curves in [8], see Remark 1 after Theorem (4) at the end of this article. Özbudak has also calculated the genera of fibre product of superelliptic curves using the “counting the differentials” and the Riemann-Hurwitz genus formula [6].

Khovanskii has given a formula for the arithmetic genus of an algebraic variety X defined in $(\mathbb{C} - 0)^n$ by a nondegenerate system of polynomial equations $f_1 = \dots = f_k = 0$ with polyhedra $\Delta_1, \dots, \Delta_k$. His method depends on choosing a toral nonsingular compactification M^n of $(\mathbb{C} - 0)^n$ and using an exact sequence of sheaves and cohomology groups of M^n [4].

Our method involves describing the projective closure. After describing the ramification indices contributed by resolving the singular points at infinity, a direct application of the Riemann-Hurwitz formula yields the genus as a function of the degrees of the hypersurfaces cutting the curve. We also show how to count the regular differential on the curve.

Our main results are Theorem (1) and Corollary (2) for the projective case and Theorem (4) and Corollary (5) for the affine cases.

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2 Projective Case

Let S denote the homogeneous coordinate ring, $k[x_0, \dots, x_n]$, of \mathbb{P}_k^n , where k is an algebraically closed field, usually \mathbb{C} . We assume that there are hypersurfaces H_1, \dots, H_r of \mathbb{P}_k^n of degrees d_1, \dots, d_r respectively such that $X_r = H_1 \cap \dots \cap H_r$. The hypersurfaces H_1, \dots, H_r correspond to homogeneous polynomials $f_1, \dots, f_r \in S$ of degrees d_1, \dots, d_r respectively.

2.1 The Hilbert Polynomial of X_r

Theorem 1 *The Hilbert polynomial $H_r(z)$ of X_r is given by the following formula*

$$H_r(z) = \varphi(z) + \sum_{m=1}^r (-1)^m \sum_{1 \leq i_1 < \dots < i_m \leq r} \varphi(z - d_{i_1} - \dots - d_{i_m}) \quad (1)$$

where

$$\varphi(z) = \frac{1}{n!} (z+1)(z+2) \cdots (z+n) = \binom{z+n}{n}.$$

Proof: From [1, Theorem 2], the Koszul complex $K(f_1, \dots, f_r)$ (which is the Koszul complex of the homomorphism $\varphi : S^r = \bigoplus_{i=1}^r S e_i \rightarrow S = k[x_0, \dots, x_n]$ such that $\varphi(e_i) = f_i$ for $1 \leq i \leq r$) is a free resolution of $S/(f_1, \dots, f_r)$. Namely, we have the following exact sequence

$$0 \rightarrow \wedge^r(S^r) \rightarrow \dots \rightarrow \wedge^2(S^r) \rightarrow S^r \rightarrow S \rightarrow S/(f_1, \dots, f_r) \rightarrow 0. \quad (2)$$

In [2], in order to grade

$$\wedge^m(S^r) = \bigoplus_{1 \leq i_1 < \dots < i_m \leq r} S e_{i_1} \wedge \dots \wedge e_{i_m} \quad (1 \leq m \leq r), \quad (3)$$

a degree $d_{i_1} + \dots + d_{i_m}$ is assigned to a basis element $e_{i_1} \wedge \dots \wedge e_{i_m}$, so that (2) is an exact sequence with maps homogeneous of degree zero. Now imposing the additive property of Hilbert polynomials on the exact sequence (2), the formula given in (1) is obtained. \square

Corollary 2 *The arithmetic genus, $g_a(X_r)$, of X_r is given by the formula*

$$g_a(X_r) = \sum_{m=1}^r (-1)^{m+n-r} \sum_{1 \leq i_1 < \dots < i_m \leq r} \varphi(-d_{i_1} - \dots - d_{i_m}). \quad (4)$$

3 Affine Case

In this section, we compute the genus of a complete intersection curve C in $\mathbb{A}_{\mathbb{C}}^{n+1}$ given by,

$$\begin{aligned} y_1^{d_1} &= (x - a_{11}) \dots (x - a_{1m}) \\ y_2^{d_2} &= (x - a_{21}) \dots (x - a_{2m}) \\ &\vdots \\ y_n^{d_n} &= (x - a_{n1}) \dots (x - a_{nm}) \end{aligned} \tag{5}$$

where $2 \leq d_1 \leq \dots \leq d_n \leq m - 1$ and all a_{ij} 's are distinct, with $a_{ij} \in \mathbb{C}$.

This is a smooth affine curve and its projective closure \overline{C} in $\mathbb{P}_{\mathbb{C}}^{n+1}$ is singular. Let \tilde{C} be a resolution of \overline{C} . The genus of C is then defined as the genus of \tilde{C} . In the following subsections we will in turn describe the projective closure of C , describe a finite map from \tilde{C} to \mathbb{P}^1 , count the ramification indices of the points of \tilde{C} under this map and finally apply the Riemann-Hurwitz formula to this map to calculate the genus.

3.1 A Finite Morphism to \mathbb{P}^1

In order to compute the genus of a nonsingular model \tilde{C} of the projective closure \overline{C} of C we first define a finite morphism from \tilde{C} to \mathbb{P}^1 .

There exists a finite morphism

$$\begin{aligned} \varphi : C &\rightarrow \mathbb{C} \\ (x, y_1, \dots, y_n) &\mapsto x \end{aligned}$$

C is embedded into \mathbb{P}^{n+1} the same way \mathbb{C} embeds into \mathbb{P}^1 . The morphism φ extends to \overline{C} algebraically by defining

$$\begin{aligned} \varphi : \overline{C} &\rightarrow \mathbb{P}^1 \\ [x : y_1 : \dots : y_n : 1] &\mapsto [x : 1] \\ [0 : y_1 : \dots : y_n : 0] &\mapsto [1 : 0] \end{aligned}$$

See also the parametrization (8) for a justification of this definition. If \tilde{C} is a resolution of \overline{C} , then C and \tilde{C} are isomorphic everywhere except at finitely many points which correspond to the points at infinity and φ extends over to \tilde{C} by sending all the points at infinity to $[1 : 0]$ as above.

Thus we have a map

$$\varphi : \tilde{C} \rightarrow \mathbb{P}^1$$

which is a morphism of degree $d_1 d_2 \dots d_n$.

3.2 Ramifications of φ

We first examine the $n = 2$ case with $d = d_1 = d_2$. Consider the curve C_1 given by the equations

$$\begin{aligned} y_1^d &= x^m + a_1 x^{m-1} + \dots + a_{m-1} x + a_m =: F_1(x) \\ y_2^d &= x^m + b_1 x^{m-1} + \dots + b_{m-1} x + b_m =: F_2(x) \end{aligned} \quad (6)$$

with $2 \leq d \leq m - 1$. For the points in the affine plane we can take x as a local parameter. When x is not equal to any of the a_{ij} 's then the ramification of φ at x is 1. When $x = a_{ij}$, then the ramification of φ at x is d . (For the general case of equation (5) the ramification at a_{ij} is $d_1 \dots \hat{d}_i \dots d_n$, where \hat{d}_i denotes that the term should be omitted.)

To examine the points at infinity choose a local parameter t with $x = 1/t$. Then we have

$$\begin{aligned} y_1^d &= x^m + a_1 x^{m-1} + \dots + a_{m-1} x + a_m \\ &= (1/t)^m + a_1 (1/t)^{m-1} + \dots + a_{m-1} (1/t) + a_m \\ &= (1 + a_1 t + \dots + a_{m-1} t^{m-1} + a_m t^m) / t^m. \end{aligned}$$

Let

$$\begin{aligned} d &= ac \\ m &= bc, \quad (a, b) = 1, \quad c \geq 1. \end{aligned} \quad (7)$$

Define a new local parameter T such that

$$T^a = t.$$

Then the above parametrization of y_1^d becomes

$$y_1^{ac} = (1 + a_1 T^a + \dots + a_m T^{abc}) / T^{abc}.$$

Similarly we have

$$y_2^{ac} = (1 + b_1 T^a + \cdots + b_m T^{abc})/T^{abc}.$$

Let $H_1(T)$ and $H_2(T)$ be power series such that $y_1^{ac} = H_1^{ac}(T^a)/T^{abc}$ and $y_2^{ac} = H_2^{ac}(T^a)/T^{abc}$. Then the points around infinity are parametrized as

$$P(\alpha_1, \alpha_2, T) = \left[\frac{T^b}{\alpha_1 T^a H_1(T^a)} : 1 : \frac{\alpha_2 H_2(T^a)}{\alpha_1 H_1(T^a)} : \frac{T^b}{\alpha_1 H_1(T^a)} \right], \quad (8)$$

where α_1 and α_2 are d -th roots of unity. Note that $H_1(0) = H_2(0) = 1$ and thus the points at infinity are of the form $[0 : 1 : \alpha_2/\alpha_1 : 0]$. In the T -plane let T_1 and T_2 be two points such that $T_2 = \lambda T_1$ where λ is an a -th root of unity. We have $T_1^a = T_2^a$ but $T_1^b \neq T_2^b$ since $(a, b) = 1$. Hence $P(\alpha_1, \alpha_2, T_1) \neq P(\alpha_1, \alpha_2, T_2)$. As T ranges in the T -plane $P(\alpha_1, \alpha_2, T)$ describes a branch of the curve at infinity. There are then $d^2/a = dc$ branches at infinity. Since there are d points at infinity, around each such point there are then c branches making the total of dc branches. Each branch corresponds to a different point on the resolution so there are dc points on the resolution corresponding to the points at infinity, i.e. the cardinality of the set $\varphi^{-1}([1 : 0]) \subset \tilde{C}$ is dc . Total ramification index for the preimage of any point under φ , i.e. the degree of φ , is d^2 . This gives a ramification index of a for each point in the resolution corresponding to the point at infinity.

In the general case when $d = d_1 = \cdots = d_n$, the total ramification index of φ is d^n , there are $d^{n-1}c$ branches at infinity each having ramification index a . This is the case for the curve define with the equations (9).

In the most general case, see equations (5), when $d = d_1 = \cdots = d_s < \cdots < d_n$ there are $d^{s-1}c$ branches at infinity each with ramification index $ad_{s+1} \cdots d_n$. In this case the cardinality of $\varphi^{-1}([a_{ij} : 1])$ is $d_1 \cdots \hat{d}_i \cdots d_n$ and the ramification index of each such point is $d_i - 1$. The total degree of φ is $d^s d_{s+1} \cdots d_n$.

Before we apply the Riemann-Hurwitz formula to find the genus we summarize our observations about the points at infinity of the curve C of equation (5) in the following corollary.

Corollary 3 *Let C be the curve in \mathbb{A}^{n+1} defined by (5), with $d_1 \leq d_2 \leq \cdots \leq d_n$. Assume that for some s the first s d_i 's are equal, i.e. $d = d_1 =$*

$d_2 = \dots = d_s < d_{s+1} < \dots < d_n$. Then the projective closure \overline{C} of the curve C in \mathbb{P}^{n+1} is the union of C and the points of the form,

$$[x : y_1 : \dots : y_s : y_{s+1} : \dots : y_n : z] = [0 : 1 : \alpha_2 : \dots : \alpha_s : 0 : \dots : 0]$$

where $\alpha_2^d = \dots = \alpha_s^d = 1$.

□

3.3 The Genus Calculation

The Riemann-Hurwitz formula for the map φ takes the form

$$\begin{aligned} g_C &= 1 - \deg \varphi + \frac{1}{2} \sum_{x \in C} (e_x - 1) \\ &= 1 - \deg \varphi + \frac{1}{2} \sum_{x \in \varphi^{-1}([*:1])} (e_x - 1) + \frac{1}{2} \sum_{x \in \varphi^{-1}([1:0])} (e_x - 1), \end{aligned}$$

where e_x denotes the ramification index.

Theorem 4 *Let C be the complete intersection curve given by,*

$$\begin{aligned} y_1^d &= (x - a_{11}) \dots (x - a_{1m}) \\ y_2^d &= (x - a_{21}) \dots (x - a_{2m}) \\ &\vdots \\ y_n^d &= (x - a_{n1}) \dots (x - a_{nm}) \end{aligned} \tag{9}$$

where $d + 1 \leq m$, and all a_{ij} 's are distinct. The genus of C is given by the formula

$$g_C = 1 - \frac{1}{2} (d - mnd + mn + c) d^{n-1} \tag{10}$$

where $c = (d, m)$.

Proof: The degree of φ is d^n . The ramification index at smooth finite points is 1 and for each point $x \in C$ for which $\varphi(x) = a_{ij}$ the ramification

index is d . There are d^{n-1} points in $\varphi(x) = a_{ij}$ and the number of a_{ij} 's is mn . This gives $\frac{1}{2}mnd^{n-1}(d-1)$ for the first summation in (10).

There are $d^{n-1}c$ points on the resolution of the projective closure of C corresponding to points at infinity. Each such point has ramification index a . This then gives $\frac{1}{2}d^{n-1}c(a-1)$ for the second summation in (10). Putting these in and simplifying gives the sought formula. \square

Remark 1: Putting in $d = 2$, $c = 1$ we recover Stepanov's formula $1 + (mn - 3)2^{n-2}$, see [8, p370, Lemma 1]. Stepanov arrives at this formula by constructing an explicit basis for the differential forms of the curve. He works over a finite field F_q of characteristic $p > 2$.

We finally give the formula for the most general case.

Corollary 5 *Let C be the complete intersection curve given by (5), with $d_1 \leq d_2 \leq \dots \leq d_n$ and with the first s d_i 's equal to d . The genus of C is given by the formula*

$$g_C = 1 - \frac{1}{2}(d - mnd + ms)d^{s-1}d_{s+1} \cdots d_n - \frac{1}{2}d^{s-1}c - \frac{md^s}{2} \sum_{i=s+1}^n d_{s+1} \cdots \hat{d}_i \cdots d_n, \quad (11)$$

where $c = (d, m)$.

\square

Remark 2: This corollary can be proved in the same way as Theorem (4). The ramification values required for the formula are given at the end of section (3.2).

Remark 3: Note that when we put $s = n$ in the above formula (11) we recover the formula (10) of Theorem (4). However this is only an algebraic phenomena since geometrically the two formulas are derived from different configurations at infinity.

3.4 Counting the Differentials

It is of interest to summarize the method of counting the regular differentials for the curve C of equation (9). See also the equations (7) for the conventions in use. For any point in the affine space let x be a local parameter and consider the regular 1-form

$$\omega_{i_1, \dots, i_\sigma}^{(j_1, \dots, j_\sigma)} = \frac{dx}{y_{i_1}^{j_1} \cdots y_{i_\sigma}^{j_\sigma}}$$

where $1 \leq \sigma \leq n$, $1 \leq i_1 < \cdots < i_\sigma \leq n$ and $1 \leq j_1, \dots, j_\sigma \leq d-1$. By checking the order of vanishings of x and y_i 's it can be shown that the form $\omega_{i_1, \dots, i_\sigma}^{(j_1, \dots, j_\sigma)}$ is regular at any point in the affine space. Let x_∞ be any point at infinity on the projective closure of C . Let ν_∞ denote the order of vanishing of a function at x_∞ . Choosing $t = 1/x$ as a local parameter around x_∞ we observe that

$$\begin{aligned} \nu_\infty(x) &= -a \\ \nu_\infty(y_i) &= -b. \end{aligned}$$

Let $\bar{\omega}_{i_1, \dots, i_\sigma}^{(j_1, \dots, j_\sigma)}$ denote the expression for $\omega_{i_1, \dots, i_\sigma}^{(j_1, \dots, j_\sigma)}$ around x_∞ . We then have

$$\nu_\infty(\bar{\omega}_{i_1, \dots, i_\sigma}^{(j_1, \dots, j_\sigma)}) = (j_1 + \cdots + j_\sigma)b - a - 1$$

and if $P(x)$ is a polynomial then $P(x)\omega_{i_1, \dots, i_\sigma}^{(j_1, \dots, j_\sigma)}$ is regular at x_∞ if and only if $\deg P(x) \leq ((j_1 + \cdots + j_\sigma)b - a - 1)/a$. We can then give a basis for the regular differential 1-forms;

$$\begin{aligned} \{ x^r \omega_{i_1, \dots, i_\sigma}^{(j_1, \dots, j_\sigma)} \mid & \sigma = 1, \dots, n, 1 \leq i_1 < \cdots < i_\sigma \leq n, \\ & 1 \leq j_1, \dots, j_\sigma \leq d-1, \\ & 0 \leq r \leq ((j_1 + \cdots + j_\sigma)b - a - 1)/a \}. \end{aligned}$$

The cardinality of this set then gives the genus of the curve C . It turns out that the required formula is

$$g(C) = \sum_{\sigma=1}^n \sum_{1 \leq i_1 < \cdots < i_\sigma \leq n} \sum_{j_1=1}^{d-1} \cdots \sum_{j_\sigma=1}^{d-1} \llbracket \frac{(j_1 + \cdots + j_\sigma)b - 1}{a} \rrbracket, \quad (12)$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function. Note that this formula now works on any algebraically closed field of any characteristic, when $a \neq 0$.

Stepanov has calculated this sum for $d = 2$ and $c = 1$ over a field of characteristic $p > 2$, [8, p372], (in that case $d = a = 2$ and $m = b$ is odd).

The sceptic reader can check the validity of equation (12) using a computer; as equation (10) suggests the genus grows exponentially as n increases so we suggest an example with a small n , for example $n = 3$. Let $d = 66$, $m = 385$. Then $a = 6$, $b = 35$ and $c = 11$. Putting these in equations (10) and (12) both gives the genus as 163 345 645.

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