Singular Holomorphic Foliations

by

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Abstract

A generalized Nash Blow-up M' with respect to coherent subsheaves of locally free sheaves is defined for complex spaces. It is shown that M' is locally isomorphic to a monoidal transformation and hence is analytic. Examples of M' are given. Applications are given to Serre's extension problem and reductive group actions. A $\mathbb{C}^*$-action on Grassmannians is defined, fixed point sets and Bialynicki-Birula decomposition are described. This action is generalized to Grassmann bundles. The Grassmann graph construction is defined for the analytic case and it is shown that for a compact Kaehler manifold the cycle at infinity is an analytic cycle. A calculation involving the localized classes of graph construction is given. Nash residue for singular holomorphic foliations is defined and it is shown that the residue of Baum-Bott and the Nash residue differ by a term that comes from the Grassmann graph construction of the singular foliation. As an application conclusions are drawn about the rationality conjecture of Baum-Bott. Pontryagin classes in the cohomology of the splitting manifold are given which obstruct an imbedding of a bundle into the tangent bundle.

Acknowledgments

I would like to thank to my thesis supervisor James B. Carrell who suggested the use of Grassmann Graph technique with residues and made several valuable comments. My thanks also go to J. King for his numerous suggestions and to L. G. Roberts for his encouragement at most needed moments.

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A. Sinan Sertöz
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This copy is processed with $\LaTeX\,2_\varepsilon$, resulting with different page numbering than the original copy. The following page numbering refers to this copy.

Chapter 1: The analyticy of the Nash blow up as given in Theorem 1 on page 8 and its application to Serre's extension problem on pages 22-24 are new but not published.

Chapter 2: The results of the $\mathbb{C}^*$-actions on Grassmann bundles appeared as *C^*-actions on Grassmann Bundles and the Cycle at Infinity,* Mathematica Scandinavica 62 (1988), 5-18.

Chapter 3: Claim of Theorem 1 on page 53 is too ambitious. The last sentence of the statement of the theorem is not touched upon until the last paragraph of the proof at page 61. The wishful claim of this last paragraph fails to materialize as can easily be seen by the examples of vector fields. The rest however is correct. Moreover the technical assumption on page 53 that the Nash blow up be smooth can be removed. For this extended and much more readable account of this theorem see *Residues of Singular Holomorphic Foliations,* Compositio Mathematica 70 (1989), 227-243.

Chapter 4: The generalized version of Bott's vanishing theorem, Theorem 1 on page 64, appeared as *On Bott's Vanishing Theorem and Applications to Singular Foliations,* Turkish Journal of Mathematics, 11 (1987), 62-67. Theorem 2 on page 67 about the existence of certain cohomology classes on the splitting manifold obstructing an embedding of a vector bundle into the tangent bundle is new but not published.

The articles mentioned above can be down loaded from
`http://www.fen.bilkent.edu.tr/~sertoz/vita.html`
either as a $\LaTeX\,2_\varepsilon$ file or as a dvi file.

The necessity of the painstaking proof techniques of this thesis remains a mystery for me to this day...

Ali Sinan Sertöz
November 1998, Ankara

Contents

CHAPTER 0
INTRODUCTION

A singular holomorphic foliation is defined as an integrable coherent subsheaf of the tangent sheaf of a complex manifold, [see BB2]. We find it promising to study coherent subsheaves of locally free sheaves in order to get a better understanding of singular holomorphic foliations. This project is carried out in Chapter 1 where a generalized Nash blow-up is defined; the context of this work is the complex analytic category. Let M be a complex analytic space with $\mathcal{G}$ a complex analytic locally free sheaf on M and $\mathcal{F}$ a complex analytic coherent subsheaf of $\mathcal{G}$. We will drop the phrase "complex analytic" from now on unless we want to emphasize it. Assume without loss of generality that M is connected and that the support of $\mathcal{F}$ is M. There is a proper closed subvariety S of M such that $\mathcal{F}$ is locally free on $M - S$. Let $\mathcal{F}$ have rank k on $M - S$ and $\mathcal{G}$ have rank n on M. For every point $x \in M - S$, $\mathcal{F}_x$ defines a k-plane in $\mathcal{G}_x$, hence a point of the Grassmann manifold $G(k, n)$ of k-planes in n-space. Let $G(k, \mathcal{G})$ denote the Grassmann bundle of k-planes in $\mathcal{G}$ over M. Thus we have a holomorphic map from $M - S$ into the Grassmann bundle $G(k, \mathcal{G})$. The topological closure of the image of this map in $G(k, \mathcal{G})$ is denoted by M' and is called the generalized Nash blow-up of M with respect to $\mathcal{F}$ and $\mathcal{G}$. In the algebraic category it is clear that M' coincides with the Zariski closure of the image and hence is algebraic. It is not however clear that M' is a complex analytic space. It is shown in Chapter 1 that M' is a complex analytic space and that in fact M' is a monoidal transformation of M whose center need not coincide with S, the singular set of $\mathcal{F}$. This generalizes works of Nobile on Nash blow-up and Rossi and Riemenschneider on blowing up coherent sheaves using the structure sheaf [N], [Ro], [Ri]. Counterexamples which yield non-analytic sets are given along with examples to the theorem. Conditions on smoothness on M' are also given. Several applications are given to Serre's extension problem and to reductive group actions.

One interesting aspect of generalized Nash blow-up is that M' comes equipped with a vector bundle, the restriction of the tautological bundle on $G(k, \mathcal{G})$, which agrees with the pull-back of $\mathcal{F}|M - S$. We then want to measure in terms of characteristic classes how much the tautological bundle differs from the pull-back of $\mathcal{F}$. For this we employ the technique of Grassmann graph construction of the algebraic category [BFM]. It must then be shown that this technique is valid in the complex analytic category. This is accomplished in Chapter 2 which starts with $\mathbb{C}^*$-actions on Grassmann manifolds. We describe the Bialynicki-Birula decomposition of this action on Grassmann manifolds and give examples. $\mathbb{C}^*$-actions are then used to describe analytically the graph construction which produces localized Chern classes. This localized class is computed for a special case at the end of the chapter as an example.

These techniques are collaborated in Chapter 3 to give a calculation of Baum-

Bott residues. First we define a Nash residue for singular holomorphic foliations for which the Nash blow-up is smooth. Then we consider those singular holomorphic foliations that are integrable images of vector bundles in the tangent bundle, i.e. let E be a vector bundle, T be the tangent bundle on M and $\Psi : \mathrm{E} \rightarrow \mathrm{T}$ be a holomorphic vector bundle map; $\Psi(\mathrm{E})$ defines a singular holomorphic foliation if it is integrable. When the Nash Blow-up of this singular holomorphic foliation is smooth the Baum-Bott residue is shown to be equal to the sum of the Nash residue and a term that is calculated by using Grassmann Graph construction on the Nash Blow-up. This result allows us to conclude that the Rationality Conjecture of Baum and Bott is true in this set up.

In chapter 4 we continue our investigation of singular holomorphic foliations by again viewing them as integrable images of vector bundles in the tangent bundle. We then ask if any vector bundle E can be imbedded into T, dropping the integrability condition on the image and requiring that Φ be injective. Several topological obstructions can be found in the literature. Here we look at the problem from a Differential Geometric point of view. The problem is pulled back to the splitting manifold M_S of T. If E imbeds into T then it is shown that certain Pontryagin classes in the cohomology ring of M_S vanish. For this we need and prove a general statement of Bott's vanishing theorem [B2]. We conclude by discussing some future research projects that follow from this work on singular holomorphic foliations.

CHAPTER 1
NASH CONSTRUCTION

0 Introduction

We will be working in the category of complex analytic spaces. A complex analytic space is locally the variety of an ideal of holomorphic functions. We will drop the phrases "complex", "analytic" or even "complex analytic" from "complex analytic space" when the reference is clear. All sheaves are complex analytic and again we will drop "complex analytic" from their names when the omission causes no ambiguity.

Let $\mathcal{F}$ be a coherent complex analytic subsheaf of a locally free complex analytic sheaf $\mathcal{G}$ on a complex space M. Outside a closed proper subvariety S of M the sheaf $\mathcal{F}$ is locally free. Assuming that $\text{rank}\mathcal{G} = n$ and $\text{rank}\mathcal{F}|M - S = k$ we can define a holomorphic map F from $M - S$ into the Grassmann bundle $G(k, \mathcal{G})$ over M. Let M' be the closure of the image of this map in the Grassmann bundle. Our contribution is to show that the map F is meromorphic in the sense of Remmert, i.e. M' is complex analytic. This is accomplished in section 3 by showing that M' is locally a monoidal transformation. The first two sections gather together some facts on coherent sheaves and monoidal transformations and set up the notation that is going to be used throughout the chapter. The third section describes the construction of M' and proves that it is analytic. We then discuss its relation to the literature, in particular to the works of Nobile, Rossi, and Riemenschneider. Smoothness of M' and conditions for M' to be globally a monoidal transformation are also discussed. Examples and counterexamples are given to demonstrate the theorem and reductive group actions are discussed as an application. In the second part of the chapter Serre's extension problem is stated and Siu's solution is given along with the necessary terminology. We then give applications to this problem which follow from the analyticity of M'.

1 Coherent Sheaves

In this section we collect together some of the facts on coherent sheaves in the forms that we are going to use them in the sequel. All sheaves are going to be complex analytic; in particular "coherent" will mean "complex analytic coherent". For further details on coherent sheaves together with the proofs of the statements of this section the reader is referred to [GH, p 695 ff], [F, p 1-34, 94-95], [C].

Let M be a complex space and let $\mathcal{O}_M$ be its structure sheaf. A sheaf of $\mathcal{O}_M$-modules will be called a sheaf of modules, the structure sheaf $\mathcal{O}_M$ being understood.

Definition: *A sheaf of modules $\mathcal{F}$ over M is called* coherent *if for every $x \in M$ there is an open neighbourhood U of x such that there exists an exact sequence*

$$\mathcal{O}_U^m \longrightarrow \mathcal{O}_U^k \longrightarrow \mathcal{F}_U \longrightarrow 0$$

for some integers m and k.

For any epimorphism of the form

$$\mathcal{O}_U^k \longrightarrow \mathcal{F}_U \longrightarrow 0$$

let R_U denote the kernel sheaf;

$$0 \longrightarrow R_U \longrightarrow \mathcal{O}_U^k \longrightarrow \mathcal{F}_U \longrightarrow 0.$$

Then by the above definition of coherence, $\mathcal{F}$ is coherent if R_U is finitely generated as a $\mathcal{O}_U$-module for all U in M. It is a classical result of Oka that if $\mathcal{F}$ is coherent then R_U is also coherent.

The most common examples of coherent sheaves arise as the sheaves of holomorphic sections of vector bundles. Such coherent sheaves are called locally free. For a locally free sheaf $\mathcal{F}$ let $s_1, \ldots, s_r$ be global holomorphic sections. Then $s_1, \ldots, s_r$ generate an analytic subsheaf which is coherent. This example can be generalized as follows; call a sheaf of modules $\mathcal{G}$ *of finite type* if for any $x \in M$ there is a neighbourhood U of x and an epimorphism

$$U^m \longrightarrow \mathcal{G}_U \longrightarrow 0$$

for some integer m. Then for a coherent sheaf any subsheaf of finite type is coherent.

If $\mathcal{G}$ is a coherent subsheaf of a coherent sheaf $\mathcal{F}$ then the quotient sheaf $\mathcal{F}/\mathcal{G}$ is also coherent. In general if there is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

such that any two of them are coherent then the third one is also coherent.

Support of a sheaf $\mathcal{F}$ is defined as

$$\mathrm{supp}\mathcal{F} = \{x \in M \mid \mathcal{F}_x \neq 0\}.$$

Assume that $\mathcal{F}$ is coherent and that $\mathrm{supp}\mathcal{F}$ is open in M. Define a function rk on M as

$$\mathrm{rk}(x) = \text{rank of } \mathcal{F}_x \text{ as an } \mathcal{O}_x - \text{module}.$$

This is an upper semi-continuous function on M. To see this let $\mathrm{rk}(p) = m$ for some $p \in M$. $\mathcal{F}$ is coherent so in particular it is locally of finite type. Hence there is an open neighbourhood U of p such that there exists an epimorphism

$$\mathcal{O}_U^m \longrightarrow \mathcal{F}_U \longrightarrow 0$$

for this m. Then there are sections $s_1, \ldots, s_m$ of $\mathcal{F}_U$ such that they generate each stalk F_x for $x \in U$. Clearly this implies that $\mathrm{rk}(x) \le m$ for all $x \in U$.

Since rk takes on nonnegative numbers, it achieves a minimum. This minimum value of rank is called the *rank* of $\mathcal{F}$ and is denoted by $\mathrm{rank}\mathcal{F}$.

Define a subset S of M as

$$S = \{x \in M \;\big|\; \mathrm{rk}(x) > \mathrm{rank}\mathcal{F}\ \}.$$

S is called the singular set of $\mathcal{F}$ and is a closed proper subvariety of M. Outside S the coherent sheaf $\mathcal{F}$ is the sheaf of sections of a vector bundle of rank $= n$ where $n = \mathrm{rank}\mathcal{F}$, i.e. $\mathcal{F}|M - S$ is locally free. For this reason $\mathrm{rank}\mathcal{F}$ is also referred to as the generic rank of $\mathcal{F}$.

The above definition of rank coincides with the more usual definition of rank which is given as

$$r(x) = \dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_x} k(x)$$

where $k(x) = \mathcal{O}_x/\mathfrak{m}_x$, $\mathfrak{m}_x$ being the maximal ideal of $\mathcal{O}_x$. The fact that $r(x) = \mathrm{rk}(x)$ follows from Nakayama's lemma:

Nakayama Lemma: *Let Λ be a finitely generated module over the ring of convergent power series $\mathbb{C}\{X_1, \ldots, X_n\}$ and let $\mathfrak{m}$ be the maximal ideal. Then*

$$a_1, \ldots, a_k \quad \text{generate } \Lambda \quad \text{iff } a_1, \ldots, a_k \quad \text{generate } \Lambda/\mathfrak{m}\Lambda.$$

For a proof of Nakayama's lemma see [GH, pp 680-681]. For the equivalence of r and rk see [Ha, p 288, (12.7.2)].

If $\mathcal{F}$ is a coherent sheaf, then it admits a local syzygy; for any point $p \in M$ there is an open neighbourhood U of p such that

$$0 \longrightarrow \mathcal{O}_U^k \longrightarrow \cdots \longrightarrow \mathcal{O}_U^m \longrightarrow \mathcal{F}_U \longrightarrow 0$$

is exact for some integers $k, ..., m$. Global syzygies however need not exist for complex analytic coherent sheaves. This difficulty is circumvented by Atiyah and Hirzebruch by passing to the real analytic category [AH]. View M as a real analytic space and let $\mathcal{A}$ be the real structure sheaf of M, i.e. $\mathcal{A}$ is the sheaf of complex valued real analytic functions. A sheaf of $\mathcal{A}$-modules on M is called a coherent sheaf of $\mathcal{A}$-modules if for every $x \in M$ there is an open neighbourhood U of x and an exact sequence

$$\mathcal{A}_U^p \longrightarrow \mathcal{A}_U^q \longrightarrow \mathcal{G}_U \longrightarrow 0$$

for some integers p and q. For any coherent sheaf $\mathcal{G}$ of $\mathcal{A}$-modules and any compact subset X of the real analytic manifold M, $\mathcal{G}_U$ has a resolution on open subsets V of X by locally free $\mathcal{A}_V$-modules, [AH, p 29, (2.6)], [BB2, p 310, (6.30)]. If M is a compact manifold then every coherent sheaf of $\mathcal{A}$-modules has a global resolution of locally free sheaves of $\mathcal{A}$-modules. For a coherent sheaf $\mathcal{F}$ of $\mathcal{O}$-modules $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{A}$ is a coherent sheaf of $\mathcal{A}$-modules. If M is compact then $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{A}$ has a global resolution

$$0 \longrightarrow H_m \longrightarrow \cdots \longrightarrow H_0 \longrightarrow \mathcal{F} \otimes \mathcal{A} \longrightarrow 0$$

where H_i are locally free sheaves of $\mathcal{A}$-modules and m is the real dimension of M. Let $\mathbf{H_i}$ be the real analytic complex vector bundle whose sheaf of real analytic sections is H_i. The Chern class of $\mathcal{F}$ is defined in terms of Chern classes of H_i's as follows; let $\mathbf{H}$ be the virtual bundle which is defined as the alternating sums of $\mathbf{H_i}$'s

$$\mathbf{H} = \sum_{i=0}^{m} (-1)^i \mathbf{H_i}.$$

Then $c(\mathcal{F}) = c(\mathbf{H}) = \prod_{i=0}^{m} c(\mathbf{H_i})^{p(i)}$ where $p(i)$ is $+1$ if i is even and -1 if i is odd.

For further details on characteristic classes of virtual bundles see [BB2]. For a proof that $c(\mathcal{F})$ depends only on $\mathcal{F}$ see [BS, p 106, (lemma 11)].

2 Monoidal Transformations

We will follow the exposition of Hironaka and Rossi [HR] to remind the reader of the terminology related to monoidal transformations, also known as Hironaka blow-ups.

We adopt the definition of meromorphic map as introduced by Remmert. Let X and Y be complex analytic spaces. A map $f : X \longrightarrow Y$ is called *meromorphic* if there exists a proper subvariety V of X such that
i) $f|X - V \longrightarrow Y$ is holomorphic,
ii) the closure in $X \times Y$ of the graph of $f|X - V$ is a complex analytic variety.

It is easy to see that this definition reduces to the definition of a meromorphic function when $X = \mathbb{C}$, the complex numbers, and $Y = \mathbb{P}^1$, the projective line.

Let $f : X \longrightarrow Y$ be a morphism of complex spaces and let I be an ideal sheaf on Y. The pull back of I under f is denoted by $f^*(I)$, and the ideal generated by $f^*(I)$ in $\mathcal{O}_X$ is denoted by $f^{-1}(I)$. If D in Y is the variety defined by I then the variety that corresponds to $f^{-1}(I)$ in X is denoted by $f^{-1}(D)$. The pair $f : X \longrightarrow Y$ and D is called the *monoidal transformation* of Y with center D if
i) the ideal sheaf $f^{-1}(I)$ on X is invertible, i.e. locally free of rank=1,
ii) if $g : Z \longrightarrow Y$ is any morphism of complex spaces having the property (i) then there is a unique morphism $h : Z \longrightarrow X$ such that $g = f \circ h$.

From the definition it is easy to see that $Y - D$ is isomorphic to $X - f^{-1}(D)$. It can also be shown that the monoidal transformation is determined by D and not X, see [HR], but we call X the monoidal transformation of Y.

To construct one such monoidal transformation of Y let U be an open subset of Y which is isomorphic to $\mathbb{C}^m$. Let $f_0, \ldots, f_n$ be holomorphic functions on U, not all identically equal to zero, and let I be the ideal generated by the f_i's. If we denote the variety of I as V then we can define a holomorphic map

$$F : U - V \longrightarrow \mathbb{P}^n$$

where

$$F(x) = [f_0(x) : \cdots : f_n(x)].$$

We wish to show that $F : U \longrightarrow \mathbb{P}^n$ is meromorphic. For this let J be the ideal generated by the functions

$$\big(x, [X_0 : \cdots : X_n]\big) \longrightarrow \big(X_i f_j(x) - X_j f_i(x)\big), \quad 0 \le i, j \le n, \;\; i \ne j$$

over the structure sheaf of $U \times \mathbb{P}^n$ where X_i are homogeneous coordinates on $\mathbb{P}^n$. The variety that is defined by J in $U \times \mathbb{P}^n$ is the closure of the graph of F. Let $\tilde{U}$ denote this closure. We have $\tilde{U} = V(J)$, hence $\tilde{U}$ is analytic and F is meromorphic.

It can be shown that $\tilde{U}$ is a monoidal transformation of U, see [HR]. The center of this transformation is $V(I)$ in U. This particular description of a monoidal transformation will be used in the next section on the Nash blow-up.

3 The Nash Construction

Let $\mathcal{F}$ be an analytic coherent subsheaf of a locally free sheaf $\mathcal{G}$ on a complex analytic space M. Let rank$\mathcal{F}$ $= k$ and rank$\mathcal{G} = n$. There is a proper analytic subvariety S of M such that $\mathcal{F}|M - S$ is locally free of rank k. Consider the Grassmann bundle $G(k, \mathcal{G})$ over M. Each fibre over $x \in M$ is the space of

k-planes in $\mathcal{G}_x$ and is therefore isomorphic to $G(k,n)$, the Grassmann space of k-planes in n-space. Define a map

$$F : M - S \longrightarrow M \times G(k,n)$$

by

$$F(x) = \big(x, [\mathcal{F}_x]\big).$$

Here $[\cdot]$ is used to denote the point that "$\cdot$ " represents in $G(k,n)$. Let M' be the topological closure of $F(M-S)$ in $M \times G(k,n)$. Let $\pi : M' \to M$ be the restriction of the natural projection $M \times G(k,n) \to M$. Then we have the

Definition: *$\pi M' \to M$ is the* Nash blow-up *of M with respect to $\mathcal{F}$ and $\mathcal{G}$.*

We will sometimes abuse language and call M' the Nash blow-up of M.

Let $\tau \to G(k,n)$ be the vector bundle on $G(k,n)$ which restricts to the tautological bundle of $G(k,n)$ on each stalk. Pull back τ to $M \times G(k,n)$ and restrict it to M'. Again use "τ" to denote this restriction and call the bundle $\tau \to M'$ the tautological bundle on M'.

If U is an open subset of M then let U' denote $\pi^{-1}(U)$ where $\pi : M' \to M$ is the Nash blow-up of M. Since the Nash blow-up is defined as the closure of the image of F, it follows that $\pi^{-1}(U)$ is the closure of $F(U - U \cap S)$ in $U \times G(k,n)$.

Theorem 1: *M' is locally a monoidal transformation of M. Consequently M' is a complex analytic space.*

Proof: U be an open neighbourhood of M such that $\mathcal{G}|U$ is trivial and $\mathcal{F}|U$ is finitely generated. Let $f_1, \ldots, f_r$ be holomorphic sections of $\mathcal{F}|U$ that generate it, where $r \geq k$. If $U \cap S = \emptyset$ then $U' \cong U$ by construction, so we consider the case when $U \cap S \neq \emptyset$. Since $\mathcal{F}|U$ is a subsheaf of the locally free sheaf $\mathcal{G}|U$ we can write each section f_i as

$$f_i = (f_{i1}, ..., f_{in}), \quad 1 \leq i \leq r$$

where f_{ij} are holomorphic functions on U. This defines an $r \times n$ matrix

$$A = \big(f_{ij}\big) \quad 1 \leq i \leq r,\ 1 \leq j \leq n.$$

The row vectors of A generate a k-plane in n-space when A is evaluated on $U - U \cap S$, since $\mathrm{rank}\mathcal{F}_x = k$ for $x \in U - U \cap S$, i.e. we have

$$\mathrm{rank}A \,\big|\, U - U \cap S = k$$

and

$$\mathrm{rank}A \,\big|\, U \cap S \leq k.$$

Let $[A(x)]$ denote the point $[\mathcal{F}_x]$ in $G(k,n) \cong G(k,\mathcal{G}_x)$ that is represented by the k-plane generated by the row vectors of $A(x)$, with $x \in U - U \cap S$.

$$\big[A(x)\big] = \big[\mathcal{F}_x\big] \in G(k,n), \quad x \in U - U \cap S.$$

Introduce an indexing set

$$B_m = \big\{(N_1, ..., N_k) \in \mathbb{Z}^n \;\big|\; 1 \le N_1 < \cdots < N_k \le m\;\big\}.$$

B_r will be used to pick k rows of A and B_n will be used to pick k columns of A. If $\mu \in B_r$ and $\beta \in B_n$ then define

$$\Delta_{\mu\beta} = \det(f_{ij}) \;\; i \in \mu, \;\; j \in \beta.$$

Let

$$A_\mu = (f_{ij}) \;\; i \in \mu, \;\; 1 \le j \le n.$$

A_μ is the $k \times n$-submatrix of A formed by choosing only the k rows of A that correspond to μ. For $\mu \in B_r$ let I_μ be the ideal generated by $\{\Delta_{\mu\beta} | \beta \in B_n\}$, the determinants of all $k \times k$-submatrices of A_μ. Since the rank of A on $U - U \cap S$ is not zero there exists a $\mu \in B_r$ for which the corresponding I_μ is not trivial. Choose and fix this μ for the rest of the argument.

$$I_\mu \ne \{0\}.$$

Recalling that $V(I_\mu)$ denotes the proper subvariety of U on which I_μ vanishes, it can be seen that $V(I_\mu)$ is not necessarily equal to $U \cap S$. This is the reason why the center of the monoidal transformation need not coincide with the singular set S of $\mathcal{F}$. On $U - V(I_\mu)$ the rank of A_μ is k and therefore it represents a k-plane defined by $\mathcal{F}_x$ in $\mathcal{G}_x$;

$$\big[A_\mu(x)\big] = \big[A(x)\big] = \big[\mathcal{F}_x\big] \in G(k,n), \quad x \in U - V(I_\mu).$$

We define a new map

$$H : U - V(I_\mu) \longrightarrow U \times G(k,n)$$

where

$$H(x) = \big(x, [A_\mu(x)]\big).$$

Notice that $U - V(I_\mu)$ and $U - U \cap S$ are dense in U. Let

$$Z = V(I_\mu) \cup (U \cap S).$$

In general Z need not be equal to $V(I_\mu)$ but since $\mathcal{F}$ is a subsheaf of a locally free sheaf we will assume that $Z = V(I_\mu)$ as in [BB2, pp 283-284].

First it will be shown that $\overline{H(U-V(I_\mu))}$ coincides with $\overline{F(U-U\cap S)}$. Clearly

$$F \bigm| U-V(I_\mu) = H \bigm| U-V(I_\mu).$$

$U-U\cap S$ is dense in U and contains $U-V(I_\mu)$. $F|U-U\cap S$ extends $F|U-V(I_\mu)$. It follows from this that

$$\overline{F(U-V(I_\mu)} = \overline{F(U-U\cap S)}.$$

Since F and H agree on $U-V(I_\mu)$ we conclude that the closure of the image of H coincides with the closure of the image of F, i.e.

$$\overline{H(U-V(I_\mu))} = \overline{F(U-U\cap S)}.$$

One important aspect of this equality is that the right hand side does not depend on the choice of μ. Therefore if the left hand side is a monoidal transformation then this will also hold for $\overline{F(U-U\cap S)}$ and will be independent of the choice of μ. To show that $\overline{H(U-V(I_\mu))}$ is a monoidal transformation order B_n in some fixed manner

$$\beta_0, \beta_1, ..., \beta_N \in B_n, \quad N = \binom{n}{k} - 1.$$

Recall that a monoidal transformation is defined as the closure of the image of T;

$$T : U - V(I_\mu) \longrightarrow U \times \mathbb{P}^N$$

where

$$T(x) = \big(x, [\Delta_{\mu\beta_0} : \cdots : \Delta_{\mu\beta_N}]\big).$$

Then $\overline{T(U-V(I_\mu))} = \tilde{U}$ is the monoidal transformation of U with center $V(I_\mu)$ and consequently $\tilde{U}$ is a complex analytic space as shown in section 2. It remains to show that $\tilde{U} \cong \overline{H(U-V(I_\mu))}$.

For this we use the Plücker imbedding Pl, which is an imbedding of $G(k,n)$ into $\mathbb{P}^N$ where $N = \binom{n}{k} - 1$. To define Pl let Λ be a $k \times n$ matrix representing a point $y \in G(k,n)$ and let $\Delta_\beta(\Lambda)$ be the determinant of the $k \times k$-submatrix of Λ determined by choosing all the rows and β-columns of Δ for $\beta \in B_n$.

$$\Delta_\beta(\Lambda) = \det(\Lambda_{ij}) \quad 1 \le i \le k,\ j \in \beta$$

Using the previously fixed ordering of B_n we define $Pl(y)$ as

$$Pl(y) = Pl([\Lambda]) = [\Delta_{\beta_0}(\Lambda) : \cdots : \Delta_{\beta_N}(\Lambda)].$$

With the aid of the Plücker imbedding define a new map

$$(id, Pl) : U \times G(k,n) \longrightarrow U \times \mathbb{P}^N$$

where

$$(id, Pl)(x, [\Lambda]) = \big(x, [\Delta_{\beta_0}(\Lambda) : \cdots : \Delta_{\beta_N}(\Lambda)]\big).$$

If $\Lambda = A_\mu$ then

$$\Delta_{\beta_i}(\Lambda) = \Delta_{\beta_i}(A_\mu) = \Delta_{\mu\beta_i} \quad 0 \leq i \leq N.$$

Therefore for $x \in U - V(I_\mu)$ we have the following equalities;

$$\begin{aligned}(id, Pl) \circ H(x) &= (id, Pl)(x, [A_\mu(x)]) \\ &= (x, [\Delta_{\beta_0}(A_\mu(x)) : \cdots : \Delta_{\beta_N}(A_\mu(x))]) \\ &= (x, [\Delta_{\mu\beta_0}(x) : \cdots : \Delta_{\mu\beta_N}(x)]) \\ &= T(x)\end{aligned}$$

i.e. we have

$$(id, Pl) \circ H = T.$$

Since Pl is an imbedding then (id, Pl) is an isomorphism of $H(U - V(I_\mu))$ onto $T(U - V(I_\mu))$. Therefore

$$(id, Pl)\overline{(H(U - V(I_\mu)))} = \overline{T(U - V(I_\mu))}.$$

Finally we list the string of equalities that we have proven; here $\tilde{U}$ is the monoidal transformation of U with center $V(I_\mu)$ and U' is the Nash blow-up of U with respect to $\mathcal{F}$ and $\mathcal{G}$.

$$\begin{aligned}\tilde{U} &= \overline{T(U - V(I_\mu))} \\ &\cong \overline{H(U - V(I_\mu))} \\ &= \overline{F(U - U \cap S)} \\ &= U'\end{aligned}$$

Hence U' is isomorphic to a monoidal transformation and in particular U' is analytic. If V is another open neighbourhood of M such that $\mathcal{G}|V$ is free and $\mathcal{F}|V$ is finitely generated with $U \cap V \neq \emptyset$, then U' and V' agree on $U \cap V$ since $\overline{F(U \cap V - U \cap V \cap S)}$ depends only on $\mathcal{F}$ and $\mathcal{G}$. $\tilde{U}$ and $\tilde{V}$ are analytic therefore U' and V' glue together to give a complex analytic space.
Hence M' is a complex analytic space. **QED**

More can be said about the nature of M' if more data is available. In chapter 3 we will be interested in the case when M' is smooth. For this we first give a definition:

Definition: *A coherent sheaf $\mathcal{F}$ of rank k on M with singular set S is called* rich *if for any $p \in S$ there exist an open neighbourhood U of p and k sections*

$s_1, ..., s_k$ *of* $\mathcal{F}|U$ *such that*
i) $s_1, ..., s_k$ *generate* $\mathcal{F}|U - U \cap S$,
ii) $s_1, ..., s_k$ *are linearly dependent on* $U \cap S$.

Examples of rich sheaves are easy to find. Complex actions of reductive groups on complex manifolds give rise to rich subsheaves of the tangent sheaf of M. Finitely generated subsheaves of locally free sheaves are rich. For a locally free sheaf subsheaves which are locally of finite type are also rich. Most foliations that will be considered will be rich. The significance of richness becomes clear in the next corollary. Let the set up be as before; $\mathcal{F}$ is a coherent subsheaf of a locally free sheaf $\mathcal{G}$ on a complex space M with S being the singular set of $\mathcal{F}$.

Corollary 1: *If* $\mathcal{F}$ *is rich then* M' *is a monoidal transformation of* M *with center* S.

Proof: The notation being as in Theorem 1 we have to establish two facts: (i) locally the center of the monoidal transformation coincides with S, and (ii) globally these two pieces glue together.
(i) This follows easily from the definition of richness.
(ii) Let U and V be open neighbourhoods in M such that U' and V' are monoidal transformations of U and V respectively, with centres $U \cap S$ and $V \cap S$ respectively. If $U \cap V \neq \emptyset$, then from the uniqueness of monoidal transformations U' and V' agree on $U \cap V$ and hence glue together. Since a monoidal transformation is uniquely determined by its centre, $(U \cup V)'$ is the monoidal transformation of $(U \cup V)$ with centre $(U \cup V) \cap S$. **QED**

The above proof showed that if $\mathcal{F}$ is rich then the centre of the monoidal transformation can be explicitly defined. Carrying on this theme we can say more; let the notation be as before, we have

Corollary 2: *If* $\mathcal{F}$ *is rich and* S *is smooth then* M' *is a complex manifold.*

Proof: If $\mathcal{F}$ is rich then M' is a monoidal transformation of M with centre S, by corollary 1. A monoidal transformation with smooth centre is smooth. **QED**

Remmert has proven the following result; if N is an analytic subset of an analytic space X and if Z is an analytic subset of $X - N$ then the closure of Z in X is an analytic space if $\dim(N) < \dim(Z)$, see [R]. It is interesting to relate this result with Theorem 1. Let X be the total space of the Grassmann bundle

$$\pi : G(k, \mathcal{G}) \longrightarrow M$$

and let N be $\pi^{-1}(S)$ which is an analytic subset of $G(k, \mathcal{G})$. Let Z be $F(M - S)$ which is also analytic and is in $G(k, \mathcal{G}) - \pi^{-1}(S)$. If $\dim(\pi^{-1}(S)) < \dim(F(M - S))$ then Remmert's theorem assures that $M' = \overline{F(M - S)}$ is analytic. In general the condition of Remmert's theorem is not satisfied. For example take rank$\mathcal{F} = 3$, rank$\mathcal{G} = 5$, $\dim(S) = 2$ and $\dim(M) = 7$; then $\dim(\pi^{-1}(S)) = 2 + 3 \cdot (5-3) = 8$ and $\dim(F(M-S)) = 7$.

As an application of theorem 1 we can retrieve Nobile's theorem on Nash blowing-up [N]. For this let us define the usual Nash construction as used by Nobile. Let X be a singular subvariety of $\mathbb{C}^n$ with dimension k. Let S be the set of singular points of X. Then the tangent sheaf TX of X is a coherent subsheaf of the tangent sheaf of $\mathbb{C}^n$. Define

$$\eta : X - S \longrightarrow X \times G(k,n)$$

where

$$\eta(x) = (x, [(TX)_x]).$$

The closure $\overline{\eta(X-S)}$ in $X \times G(k,n)$ is called the Nash Blowing-up of X and is denoted by *, see [N].

Theorem (Nobile): *A Nash Blowing-up is locally a monoidal transformation (with centre a suitable ideal).*

Proof: TX is a coherent subsheaf of the tangent sheaf T of $\mathbb{C}^n$ with

$$\text{supp}TX = X.$$

Let

$$\mathcal{F} = TX \mid X, \quad \mathcal{G} = T \mid X.$$

Then the generalized Nash blow-up of X with respect to $\mathcal{F}$ and $\mathcal{G}$ as in theorem 1 is locally a monoidal transformation. **QED**

H. Rossi has proven that for any coherent sheaf $\mathcal{F}$ on M with singular set S, there exists an analytic space N with a proper map

$$\psi : N \longrightarrow M$$

such that
(i) $\psi : N - \psi^{-1}(S) \to M - S$ is biholomorphic and
(ii) $\psi^*\mathcal{F}$ is locally free modulo torsion.
Rossi constructs this N as follows, see [Ro]. Let $p \in S$. Since $\mathcal{F}$ is coherent there exists an open neighbourhood U of p such that there is an exact sequence

$$\mathcal{O}_U^m \longrightarrow \mathcal{O}_U^n \longrightarrow \mathcal{F}_U \longrightarrow 0$$

for some integers m and n, where $\mathcal{O}$ is the structure sheaf of M. Let I be the image sheaf of the map

$$\mathcal{O}_U^m \longrightarrow \mathcal{O}_U^n.$$

Then I is a coherent subsheaf of $\mathcal{O}_U^n$ and is locally free of rank $n-k$ on $U-U\cap S$ where k is rank$\mathcal{F}$. For each point $x \in U - U \cap S$, I_x defines an $(n-k)$-plane in the n-space of $\mathcal{O}_U^n$. Define a map

$$\eta : U - U \cap S \longrightarrow U \times G(n-k, n)$$

where

$$\eta(x) = (x, [I_x]).$$

Then Rossi defines N as, see [Ro],

$$N \bigm| U = \overline{\eta(U - U \cap S)} \ \text{ in } \ U \times G(n-k, n).$$

Theorem (Rossi): *η is a meromorphic map.*

Proof: We have to show that $N|U$ is a complex analytic space. $N|U$ is the generalized Nash blow-up of U with respect to $I|U$ and $\mathcal{O}_U^n$, hence is analytic by theorem 1. **QED**

That $N|U$'s glue together to give a complex analytic space N follows from the uniqueness of monoidal transformations.

Later O. Riemenschneider has shown that if N_1 is another analytic space with a proper map

$$\chi : N_1 \longrightarrow M$$

such that
(i) $\chi : N_1 - \chi^{-1}(S) \to M - S$ is biholomorphic and
(ii) $\chi^* \mathcal{F}$ is locally free modulo torsion,
then there is a unique holomorphic map $\omega : N_1 \to N$ such that $\chi = \psi \circ \omega$, see [Ri]. He also showed that if $\mathcal{F}$ is an ideal sheaf then N coincides with a monoidal transformation. Our Nash construction shows that M' is always locally a monoidal transformation. The universal property of M' is an interesting problem which we propose to study elsewhere.

In the next section we will give several examples of the Nash construction.

4 Examples

1) Consider the $\mathbb{C}^*$-action on $\mathbb{P}^n$;

$$\lambda \cdot [x_0 : \cdots : x_n] = [x_0 : \lambda^{a(1)} x_1 : \cdots : \lambda^{a(n)} x_n]$$

where $\lambda \in \mathbb{C}^*$, $a(i)$ are integers with $0 < a(1) < \cdots < a(n)$, and $[x_0 : \cdots : x_n]$ are homogeneous coordinates of $\mathbb{P}^n$. The fixed points of this $\mathbb{C}^*$-action are

$$[1 : 0 : \cdots : 0], [0 : 1 : \cdots : 0], ..., [0 : 0 : \cdots : 1].$$

The orbits of this action define a singular holomorphic foliation whose set of singularities is the fixed point set of the $\mathbb{C}^*$-action. This follows since the isotropy groups are finite.

The Nash blow-up of this singular holomorphic foliation is a smooth manifold. To see this let us choose Euclidean coordinates around the fixed point $[1 : 0 : \cdots : 0]$;

$$X_1 = x_1/x_0, ..., X_n = x_n/x_0.$$

The above $\mathbb{C}^*$-action has the following form with respect to these coordinates;

$$\lambda \cdot (X_1, ..., X_n) = (\lambda^{a(1)} X_1, ..., \lambda^{a(n)} X_n).$$

In this coordinate patch the fixed point set is the origin $\underline{0} = (0, ..., 0)$. Any $X = (X_1, ..., X_n) \in \mathbb{C}^n$ defines a holomorphic curve

$$\lambda \longrightarrow \lambda \cdot X = (\lambda^{a(1)} X_1, ..., \lambda^{a(n)} X_n).$$

The direction of this curve at X, i.e. when $\lambda = 1$, is $[a(1)X_1 : \cdots : a(n)X_n]$ in $\mathbb{P}^n$. This is the image of X under the Nash construction; to see why, let $\mathbb{P}(\mathrm{T})$ be the projectivized tangent bundle of $\mathbb{C}^n$,

$$\mathbb{P}(\mathrm{T}) = \mathbb{C}^n \times \mathbb{P}^{n-1}.$$

Define a section S of $\mathbb{P}(\mathrm{T})$ as

$$S : \mathbb{C}^n - \underline{0} \longrightarrow \mathbb{P}(\mathrm{T})$$

where

$$S(X) = (X, [a(1)X_1 : \cdots : a(n)X_n]).$$

Then the Nash blow-up of this coordinate patch with respect to the above $\mathbb{C}^*$-action is the closure of $S(\mathbb{C}^n - \underline{0})$ in $\mathbb{P}(\mathrm{T})$. Let M_0 denote this closure. To show that M_0 is a complex manifold we construct the following coordinate maps; let $e = [e_1 : \cdots : e_n]$ be a point in $\mathbb{P}^{n-1}$ and let $t \in \mathbb{C}$. Define a map

$$\Lambda_i : \mathbb{P}^{n-1} \times \mathbb{C} \longrightarrow M_0$$

where

$$\Lambda_i(e, t) = \big((\frac{te_1}{a(1)e_i}, ..., \frac{te_n}{a(n)e_i}), e \big)$$

for $e_i \neq 0$ and $i = 1, ..., n$. Different Λ_i's patch together to define a holomorphic coordinate system. Hence M_0 is a complex manifold. Let

$$\pi : M_0 \longrightarrow \mathbb{C}^n$$

be the natural projection. Then the fibre above the singular set $\underline{0}$ of the singular foliation is $\pi^{-1}(\underline{0}) = \mathbb{P}^{n-1}$.

Similarly we can construct M_i by blowing up that coordinate patch of $\mathbb{P}^n$ with $\{x_i \neq 0\}$. All these M_i's are smooth and they glue together to form a complex manifold M. It is possible to give defining equations for M_i. If for example $\big((x_1, ..., x_n), [y_1 : \cdots : y_n]\big)$ are coordinates for $\mathbb{P}(\mathrm{T})|U_0$ then M_0 is defined by the equations

$$a(j)y_i x_j = a(i)y_j x_i, \quad i, j = 1, ..., n, \;\; i \neq j.$$

The isomorphism between the Nash blow-up M_0 and the monoidal transformation of $\{x_0 \neq 0\}$ with centre $[1 : 0 : \cdots 0]$ is given by the map

$$\Upsilon : \mathbb{P}(\mathrm{T}) \longrightarrow \mathbb{P}(\mathrm{T})$$

where

$$\Upsilon\big((x_1, ..., x_n), [y_1 : \cdots : y_n]\big) = \big((x_1, ..., x_n), [a(1)y_1 : \cdots : a(n)y_n]\big).$$

2) In this example we will define a rank 2 singular foliation on $\mathbb{P}^4$ and construct its Nash blow-up. Start with two $\mathbb{C}^*$-actions on $\mathbb{P}^4$ defined as

$$\lambda \cdot [x_0 : \cdots : x_4] = [\lambda^{c_{10}} x_0 : \cdots : \lambda^{c_{14}} x_4]$$

and

$$\lambda \cdot [x_0 : \cdots : x_4] = [\lambda^{c_{20}} x_0 : \cdots : \lambda^{c_{24}} x_4]$$

where $\lambda \in \mathbb{C}^*$, $[x_0 : \cdots : x_4] \in \mathbb{P}^4$ and $c_{ij} \in \mathbb{Z}$, $i = 1, 2$, $j = 0, ..., 4$. Define a 2×4 matrix C_k, for $k = 0, ..., 4$, as follows

$$C_k = (c_{ij} - c_{ik}), \quad 1 \leq i \leq 2, \;\; 0 \leq j \leq 4, \;\; j \neq k.$$

Choose c_{ij} such that 1) all the entries of C_k are nonzero and 2) the determinants of all the 2×2 minors of C_k are nonzero. To describe a rank 2 singular foliation on $\mathbb{P}^4$ choose Euclidean coordinate patches

$$U_i = \{x_i \neq 0\}, \; i = 0, ..., 4.$$

In U_0 let

$$X_1 = x_1/x_0, ..., X_4 = x_4/x_0.$$

On U_0 let a_{ij} denote the entries of the matrix C_0

$$C_0 = (a_{ij}), \quad 1 \leq i \leq 2, \;\; 0 \leq j \leq 4.$$

Note that by the above conditions on C_k the entries a_{ij} are nonzero, $a_{ij} = c_{ij} - c_{i0}$, $1 \leq i \leq 2$, $0 \leq j \leq 4$. On U_0 the above $\mathbb{C}^*$-actions take the form

$$\lambda \cdot (X_1, ..., X_4) = (\lambda^{a_{11}} X_1, ..., \lambda^{a_{14}} X_4)$$

and

$$\lambda \cdot (X_1, ..., X_4) = (\lambda^{a_{21}} X_1, ..., \lambda^{a_{24}} X_4)$$

At any point $X = (X_1, ..., X_4)$ for each $\mathbb{C}^*$-action the direction of the orbit passing through X defines vector fields

$$V_1(X) = (a_{11} X_1, ..., a_{14} X_4)$$

and

$$V_2(X) = (a_{21} X_1, ..., a_{24} X_4).$$

Notice that V_i, $i = 1, 2$, is defined for all $X \in U_0 \cong \mathbb{C}^4$. $V_1(X)$ and $V_2(X)$ together generate a vector subspace of the tangent space of U_0 at the point $X \in U_0$. Let

$$V(X) = (a_{ij} X_j) \;\; 1 \leq i \leq 2, \;\; 1 \leq j \leq 4,$$

and

$$S_0 = \{ X \in U_0 \;\big|\; \operatorname{rank} V(X) < 2 \;\}.$$

On $U_0 - S_0$, V_1 and V_2 span an integrable 2-subbundle of the tangent bundle. To construct the Nash blow-up define a map

$$F : U_0 - S_0 \longrightarrow U_0 \times G(2,4)$$

where

$$F(X) = \big(X, [V(X)]\big).$$

The closure $\overline{F(U_0 - S_0)}$ of $F(U_0 - S_0)$ in $U_0 \times G(2,4)$ is the Nash blow-up of U_0 with respect to the above $\mathbb{C}^*$-actions˙

Let $\pi : M_0 \to U_0$ be the usual projection. Then as in the previous example it can be shown that
1) $M_0 - \pi^{-1}(\underline{0})$ is a complex manifold.
$V_1(X)$ and $V_2(X)$ generate a coherent subsheaf $\mathcal{F}$ of the tangent sheaf T_0 of $U_0 - (\underline{0})$. $V_1(X)$ and $V_2(X)$ are linearly dependent along the coordinate axes and by the definition of rich sheaves $\mathcal{F}$ is a rich coherent sheaf with singularities

along the axes. Since the singular set is smooth, by corollary 2 of section 3 the Nash bşow-up, $M_0 - \pi^{-1}(\underline{0})$, is also smooth.
2) $\pi : M_0 - \pi^{-1}(S_0) \to U_0 - S_0$ is an isomorphism.
This follows from the above explanations by observing that S_0 consists of the coordinate axes in U_0.
3) For $X \in S_0 - \underline{0}$, $\pi^{-1}(X) \cong \mathbb{P}^2$.
4) $\pi^{-1}(\underline{0})$ is isomorphic to four copies of $\mathbb{P}^2$ in $G(2,4)$, which touch each other at a single point. Thus four planes each touching each of the other three at a single point give six singularities of M_0.

These last two assertions follow if a matrix representation for elements of $U_0 \times G(2,4)$ is used in calculating the closure of $F(U_0 - S_0)$. In particular $\pi^{-1}(\underline{0})$ is obtained by observing the fibres in M_0 above the coordinate axes; the four copies of $\mathbb{P}^2$ in $\pi^{-1}(\underline{0})$ are each contributed by a coordinate axis at the origin. The fact that they touch each other is again found by using the matrix representation and calculating the limits. These calculations are straightforward and are omitted.

We can similarly construct $M_1, ..., M_4$ each of which will satisfy similar conditions 1-4 as above. They glue together to give an analytic space M with 30 isolated singularities.

3) An example used in [CS1] can be adopted to describe a graph construction which gives rise to a nonanalytic set. Let $\mathbb{Z}$ act on $\mathbb{C}^2$

$$\mathbb{Z} \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

where

$$n \cdot (z_1, z_2) = (2^n z_1, 2^n z_2).$$

Let $H = \mathbb{C}^2 - \{(0,0)\}/\mathbb{Z}$ and define

$$\pi : H \longrightarrow \mathbb{P}^1$$

as

$$\pi(< z, w >) = [z : w],$$

where $< z, w >$ denotes the equivalence class of $(z, w) \in \mathbb{C}^2$ in H. There is a $\mathbb{C}^*$-action on H defined as follows

$$\mathbb{C}^* \times H \longrightarrow H$$

where

$$\lambda \cdot < z, w > = < \lambda z, w > .$$

Similarly we have a $\mathbb{C}^*$-action on $\mathbb{P}^1$ defined as

$$\mathbb{C}^* \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

where

$$\lambda \cdot [x_0 : x_1] = [\lambda x_0 : x_1].$$

π is equivariant with respect to this $\mathbb{C}^*$-action. Let T denote the preimage of $[0:1]$, i.e.

$$T = \pi^{-1}\big([0:1]\big).$$

T is the fixed point set of the $\mathbb{C}^*$-action on H.
$H - T$ is isomorphic to $(\mathbb{C}^*/\mathbb{Z}) \times \mathbb{C}$ under the map which sends $< z, w >$ to $(< z, 0 >, w/z)$. Consider the bundle projection

$$pr : H - T \longrightarrow \mathbb{C}^*/\mathbb{Z}$$

where

$$pr\big(< z, w >\big) \longrightarrow < z, 0 > .$$

Let Γ denote the closure of the graph of pr in $H \times (\mathbb{C}^*/\mathbb{Z})$, let $\Gamma_{<z,0>}$ denote the points over $< z, 0 >$ in Γ, i.e. if

$$p_2 : H \times (\mathbb{C}^*/\mathbb{Z}) \longrightarrow \mathbb{C}^*/\mathbb{Z}$$

is the natural projection then $\Gamma_{<z,0>} = p_2^{-1}(< z, 0 >)$. It can be shown that

$$\begin{aligned} \Gamma_{<z,0>} \quad &= \quad \big\{(< z, t >, < z, 0 >) \in H \times (\mathbb{C}^*/\mathbb{Z}) \bigm| t \in \mathbb{C}\big\} \\ &\quad\ \cup \big\{(< 0, \lambda >, < z, 0 >) \bigm| \lambda \in \mathbb{C}^*\big\} \end{aligned}$$

It is clear how to obtain the first component. To obtain the second component consider

$$\lim_{n\to\infty}(< z, 2^n\lambda >, < z, 0 >) = \lim_{n\to\infty}(< 2^{-n}z, \lambda >, < 2^{-n}z, 0 >).$$

But since $< 2^{-n}z, 0 >=< z, 0 >$ on $\mathbb{C}^*/\mathbb{Z}$, this final limit is $(< 0, \lambda >, < z, 0 >)$ which is our second component. If Γ is a subvariety then it must be irreducible. But the second component

$$\big\{(< 0, \lambda >, < z, 0 >) \bigm| \lambda \in \mathbb{C}^*\big\}$$

is a closed subvariety of $H \times (\mathbb{C}^*/\mathbb{Z})$ whose dimension is equal to the dimension of Γ. This composition shows that Γ is not analytic.

Recall that our Nash construction is also a graph closure. In the above example the map pr can not be meromorphically extended across T whereas the data that we supply for the Nash construction guarantees that the map constructed there is always meromorphic.

Another such example using Hopf manifolds is given in [G, p29, eg4].

4) A simple example of nonanalytic graph closure can be constructed by considering a holomorphic function

$$f : \mathbb{C}^* \longrightarrow \mathbb{C}$$

with an essential singularity at the origin. Since by Picard's theorem f attains every value infinitely many times in any neighbourhood of the singularity, the closure Γ of the graph of f in $\mathbb{C} \times \mathbb{C}$ can not be analytic. Otherwise we can intersect the graph Γ with $\mathbb{C} \times \{\gamma\}$ for any $\gamma \in \mathbb{C}$ and we should get an analytic variety as the intersection. But by Picard's theorem $(\mathbb{C} \times \{\gamma\}) \cdot \Gamma$ will have infinitely many isolated points, hence is not an analytic variety. This contradiction shows that Γ is not analytic.

5) Let G be a connected compact Lie group and M a complex manifold. Let

$$\Psi : G \times M \longrightarrow M$$

be a C^∞ action of G by means of biholomorphisms. If V is a real vector field induced by a 1-parameter subgroup of G and J is the complex structure tensor of M, then $V - iJV = W$ is a holomorphic vector field and the fixed point set of Ψ is a complex submanifold which is the set of common zeros of all such W, [CS1, page 50]. These W's generate a coherent integrable subsheaf $\mathcal{F}$ of the tangent sheaf T of M, i.e. $\mathcal{F}$ is closed under bracket operation. By Richardson's theory G has a principal orbit type, i.e. on an open dense subset U of M all orbits have the same rank, see [Rc]. The closure of an orbit picks up orbits of smaller ranks. Singular set of $\mathcal{F}$ consists of the union of all orbits whose ranks are less than the rank of the principal orbit. Let $\mathcal{G}$ be the integrable locally free subsheaf of $T|U$ defined by the principal orbits of G in U. Then

$$\mathcal{F} \,\Big|\, U = \mathcal{G}.$$

Hence the Nash construction defined by using the principal orbits coincide with the Nash construction of M with respect to $\mathcal{F}$ and T, and consequently is analytic.

6) Consider a $\mathbb{C}^*$-action on $\mathbb{C}^n$ as

$$\mathbb{C}^* \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

where

$$\lambda \cdot (X_1, ..., X_n) = (\lambda^{a(1)} X_1, ..., \lambda^{a(n)} X_n)$$

with $a(i) \in \mathbb{Z}$. The Nash blow-up of $\mathbb{C}^n$ with respect to the orbits of this action is a smooth submanifold M of $\mathbb{C}^n \times \mathbb{P}^{n-1}$. Let

$$\pi : M \longrightarrow \mathbb{C}^n$$

be the usual projection. We can define a $\mathbb{C}^*$-action on M which extends the $\mathbb{C}^*$-action on $\pi^{-1}(\mathbb{C}^n - \underline{0})$;

$$\mathbb{C}^* \times M \longrightarrow M$$

where for $\lambda \in \mathbb{C}^*$ and $p \in M$ define $\lambda \cdot p$ as follows:
1) If $p = \big((X_1, ..., X_n), [a(1)X_1 : \cdots : a(n)X_n]\big)$, then
$\lambda \cdot p = \big((\lambda^{a(1)}X_1, ..., \lambda^{a(n)}X_n), [a(1)\lambda^{a(1)}X_1 : \cdots : a(n)\lambda^{a(n)}X_n]\big)$.
Clearly $\lambda \cdot p \in M$.
2) If $p = \big(\underline{0}, [X_1 : \cdots : X_n]\big)$, then $\lambda \cdot p = \big(\underline{0}, [\lambda^{a(1)}X_1 : \cdots : \lambda^{a(n)}X_n]\big)$.
This action is the restriction to M of the $\mathbb{C}^*$-action of $\mathbb{C}^n \times \mathbb{P}^{n-1}$ which is defined as follows

$$\lambda \cdot \big((x_1, ..., x_n), [e_1 : \cdots : e_n]\big) = \big((\lambda^{a(1)}x_1, ..., \lambda^{a(n)}x_n), [\lambda^{a(1)}e_1 : \cdots : \lambda^{a(n)}e_n]\big)$$

where $\lambda \in \mathbb{C}^*$, $(x_1, ..., x_n) \in \mathbb{C}^n$ and $[e_1 : \cdots : e_n] \in \mathbb{P}^{n-1}$.

In this example the $\mathbb{C}^*$-action on $\mathbb{C}^n$ is lifted up to the Nash blow-up that is defined by the $\mathbb{C}^*$-action. It can be conjectured that reductive group actions can be lifted to the Nash blow-up that is defined in the previous example.

7) Let E and F be vector bundles on a complex space M Let $\text{rankE} = r$ and $\text{rankF} = n$. Let

$$\Phi : \mathrm{E} \longrightarrow \mathrm{F}$$

be a vector bundle map. There exists a proper subvariety S of M such that

$$\text{rank}\Phi \,\big|\, M - S = k$$
$$\text{rank}\Phi \,\big|\, S < k$$

where k is some integer not greater than r. Then we can define a meromorphic map

$$\mathbf{F} : M \longrightarrow M \times G(k, \mathrm{F})$$

as

$$\mathbf{F}(x) = (x, [x, [\Phi(x)]).$$

The proof that $\mathbf{F}$ is meromorphic is quite analogous to the proof of theorem 1 therefore we do not repeat it here. If M is a complex manifold and F is its tangent bundle then $\Phi(\mathrm{E})$ defines a distribution. If in addition to this, $\Phi(\mathrm{E})$ is integrable then it defines a singular holomorphic foliation whose singularity set is S, i.e. $M - S$ is foliated with k-dimensional leaves and for any $x \in M - S$ the tangent space of the leaf that passes through x is $\Phi_x(\mathrm{E})$. We will return to this approach to singular foliations in Chapter 3 and 4.

5 Serre's Extension Problem

In this section we will give an application of our theorem to Serre's problem on extending coherent sheaves. In 1966 Serre posed his famous extension problem, [S]. Let M be a complex analytic space with a closed analytic subset Z and a coherent analytic sheaf $\mathcal{F}$ defined on $M-Z$. If $i : M-Z \to M$ is the usual inclusion then $i_*\mathcal{F}$ denotes the direct image of $\mathcal{F}$, i.e. for any open subset of M the sheaf of sections of the sheaf $i_*\mathcal{F}$ on U, $\Gamma(U, i_*\mathcal{F})$, is defined to be equal to $\Gamma(U-Z, \mathcal{F})$. Serre's problem is to determine if $i_*\mathcal{F}$ is coherent. This problem has been successfully attacked by Siu, Trautmann and Thimm, see [ST], [T]. Serre himself proved the following, [S];

Theorem (Serre): *If M is normal, $\mathcal{F}$ is torsion free and codim$(Z) \geq 2$ then the following are equivalent:*
i) $i_*\mathcal{F}$ is coherent.
ii) There is an analytic coherent sheaf E on M which extends $\mathcal{F}$.
iii) For all $p \in Z$, there is an open neighbourhood U of p such that for all $x \in U-Z$ the image of $\Gamma(U-Z, \mathcal{F})$ generates $\mathcal{F}_x$ as an $\mathcal{O}_x$-module, where $\mathcal{O}$ is the structure sheaf of M.

Obviously (i) implies (ii). It is surprising however that (ii) does not imply (i) without the assumptions of the theorem; for example $i_*\mathcal{O}_{M-Z}$ is not coherent, [S].

Definition: *If $i_*\mathcal{F}$ is coherent then $\mathcal{F}$ is called* extendible.

The set up being as above let $\mathcal{G}$ be a locally free sheaf on M and let $\mathcal{F}$ be a coherent subsheaf of $\mathcal{G}$ on $M-Z$. Let M' be the Nash Blow-up of M with respect to $\mathcal{F}$ and $\mathcal{G}$. Since $\mathcal{F}$ is not defined everywhere theorem 1 of section 3 does not directly apply. In this case we have the following theorem.

Theorem 2: *Assume that $\mathcal{F}$ is torsion free, M is normal and codim$(Z) \geq 2$. Then M' is analytic iff $\mathcal{F}$ is extendible as a coherent subsheaf of $\mathcal{G}$.*

Proof: If $\mathcal{F}$ is extendable, i.e. $i_*\mathcal{F}$ is a coherent subsheaf of $\mathcal{G}$, then M' is the Nash Blow-up of M with respect to $i_*\mathcal{F}$ and $\mathcal{G}$, and hence is analytic by theorem 1 of section 3. Assume then that M' is analytic. Let rank$\mathcal{F} = k$ and rank$\mathcal{G} = n$. Then let $G(k, \mathcal{G}) \to M$ be the Grassmann bundle of k-planes in $\mathcal{G}$. Let $\tau \to G(k, \mathcal{G})$ be that vector bundle which restricts to the tautological vector bundle of $G(k, n)$ on each fibre of $G(k, \mathcal{G})$. Use the same notation $\tau \to M'$ to denote the restriction of this bundle to M'. On M' we have the following short exact sequence of bundles:

$$0 \longrightarrow \tau \longrightarrow \mathbb{C}^n \longrightarrow Q \longrightarrow 0 \qquad (*)$$

where Q is the quotient bundle $\mathbb{C}^n/\tau$. Let $\pi : M' \to M$ be the usual projection.

On $M - Z$ there is a short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow K \longrightarrow 0$$

where K is the quotient sheaf $\mathcal{G}/\mathcal{F}$. This sequence pulls back by π^* to the sequence of sheaves corresponding to the sequence of bundles $(*)$ on $M' - \pi^{-1}(Z)$. Hence on M' the sheaf of sections of τ, which we denote by $\underline{\tau}$ is a coherent sheaf that extends $\pi^*\mathcal{F}$. This extension lies in $\underline{\mathbb{C}}^n = \pi^*\mathcal{G}$. Since π is a proper map $\pi_*\underline{\tau}$ is a coherent sheaf on M and since $\underline{\tau}$ lies in $\pi_*\mathcal{G}$ the coherent sheaf $\pi_*\underline{\tau}$ lies in $\mathcal{G}$. Then by Serre's theorem $i_*\mathcal{F}$ is coherent. Hence $\mathcal{F}$ is extendable. **QED**

This result not only tells us when M' is analytic but also provides a solution to Serre's extension problem when $\mathcal{F}$ is torsion free. In general if $\mathcal{F}$ has torsion then we can ask if there exists an analytic coherent subsheaf E of $\mathcal{G}$ which extends $\mathcal{F}$. In this case E need not be equal to $i_*\mathcal{F}$.

Corollary 3: *Let M be a complex analytic space with a proper analytic subset Z. Let $\mathcal{G}$ be a locally free sheaf on M and $\mathcal{F} \to M - Z$ be a coherent subsheaf of $\mathcal{G}|M - Z$. Then there exists an analytic coherent subsheaf $E \to M$ which extends $\mathcal{F}$ iff the Nash construction M' of M with respect to $\mathcal{F}$ and $\mathcal{G}$ is analytic.*

Proof: Using the notation of the above theorem if M' is analytic then $E = \pi_*\underline{\tau}$ extends $\mathcal{F}$. If there exists a coherent E which extends $\mathcal{F}$ then the Nash Blow-up of M with respect to $\mathcal{F}$ and $\mathcal{G}$ is precisely the Nash Blow-up of M with respect to E and $\mathcal{G}$ and is therefore analytic by theorem 1 of section 3. **QED**

Other solutions to Serre's problem provides us with further criteria in deciding when M' is analytic. Let us revive our set up. M is an analytic space with the proper closed analytic subset Z. There is a locally free sheaf $\mathcal{G}$ of M and a coherent sheaf $\mathcal{F}$ of $M - Z$. Let $\mathcal{F}$ be a subsheaf of $\mathcal{G}$ on $M - Z$. Let $\operatorname{rank}\mathcal{F} = k$, $\operatorname{rank}\mathcal{G} = n$ and S be the singular set of $\mathcal{F}$. Then on each point x of $Z \cup S$, $\mathcal{F}_x$ defines a k-plane in $\mathcal{G}_x$ and this defines an imbedding of $M - Z \cup S$ into $G(k, \mathcal{G})$. The closure of the image of this imbedding is denoted by M', and $\pi : M' \to M$ is the usual projection. We know that $M' - \pi^{-1}(Z)$ is analytic. However $\mathcal{F}$ need not extend coherently over Z and we cannot say much about M' in general. Therefore we look for conditions on $\mathcal{F}$ which make it possible to extend it coherently over Z. If $\mathcal{F}$ extends coherently then M' is analytic. The following theorem is due to Siu and Trautmann. For the proof see [ST].

Theorem (Siu, Trautmann): *Let M be a complex space and D an open subset which is strongly r-concave at a point $x_0 \in M$. Let $\mathcal{G}$ be a coherent analytic sheaf on M and $\mathcal{F}$ a coherent analytic sheaf on D. Let $\mathcal{F}$ be a subsheaf of $\mathcal{G}|D$. If the r-th relative gap sheaf of $\mathcal{F}$ in $\mathcal{G}$ is equal to $\mathcal{F}$, i.e. $\mathcal{F}_r = \mathcal{F}$, then $\mathcal{F}$ can be extended coherently to an open neighbourhood of x_0 as a subsheaf of $\mathcal{G}$.*

Corollary 4: *Let M and D be as above, $\mathcal{G}$ be a locally free sheaf on M, $\mathcal{F}$ a coherent sheaf defined on D. Let $\mathcal{F}$ be subsheaf of $\mathcal{G}|D$. If $\mathcal{F}_r = \mathcal{F}$ then M' is analytic.* **QED**

Let us briefly describe the terms used. A twice differentiable real valued function f is said to be strongly r-convex on a subset D of $\mathbb{C}^n$ if at every point of D the hermitian matrix

$$\left(\frac{\partial^2 f}{\partial z_i \partial \overline{z_j}}\right)$$

has at least $n-r+1$ positive eigenvalues. An open subset U of a complex space M is strongly r-concave at a point $x_0 \in M$ if there is an open neighbourhood V of x_0 on which there is a strongly r-convex function f such that

$$f(x_0) = 0 \quad \text{and} \quad U \cap V = \{y \in V \ \big| \ f(y) > 0\ \}.$$

For a subvariety A of M the gap sheaf $\mathcal{F}[A]$ of $\mathcal{F}$ in $\mathcal{G}$ with respect to A is the sheaf defined by the presheaf

$$U \mapsto \{s \in \Gamma(U, \mathcal{G}) \ \big| \ s|U - A \in \Gamma(U - A, \mathcal{F})\}.$$

The r-th relative gap sheaf $\mathcal{F}_r$ of $\mathcal{F}$ in $\mathcal{G}$ is the sheaf defined by the presheaf

$$U \mapsto \varinjlim \{\Gamma(U, \mathcal{F}[A]) \ \big| \ A \ \text{ is a subvariety of } \ U,\ \dim A \leq r\}.$$

Finally before we close this chapter we give an alternate description of Nash construction, borrowing an idea of Giraud [Gr]. Let $\mathcal{F}$ be a coherent sheaf on M with S being the singular set of $\mathcal{F}$. Let $\mathcal{O}$ be the structure sheaf of M. Recalling that $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules construct the sheaf

$$\mathrm{Proj}(\Lambda^k \mathcal{F}) \longrightarrow M.$$

On $M - S$ there is a natural imbedding of M into this sheaf. The closure of the image of this imbedding can be called the intrinsic Nash construction on M. See also Thimm [T] where he mentions Plückerian coordinates of $\mathcal{F}$.

CHAPTER 2
GRASSMANN GRAPH CONSTRUCTION

0 Introduction

This chapter describes the Grassmann Graph construction of MacPherson in the analytic category using $\mathbb{C}^*$-actions. The details of the algebraic case can be found in [BFM].

In section 1 we summarize the decomposition theorem of Bialynicki-Birula in the compact Kaehler case, [BBc], [CS1]. Section 2 describes a $\mathbb{C}^*$-action on Grassmann manifolds and gives the corresponding Bialynicki-Birula decomposition. Examples are given in the next section. In section 4 this $\mathbb{C}^*$-action is carried on to Grassmann bundles and $\mathbf{Z}_\infty$, the cycle at infinity corresponding to a bundle morphism, is defined. It is shown that in the compact Kaehler case $\mathbf{Z}_\infty$ is an analytic cycle. The graph construction is finally accomplished in section 5. Examples are given in section 6.

1 Bialynicki-Birula Decomposition

The references for this section are [BBc] for the algebraic case and [CS1] for the complex case. There is also a clear summary in [CS2, section Ic].

Let M be a compact Kaehler manifold with a $\mathbb{C}^*$-action on it. Let this $\mathbb{C}^*$-action have nontrivial fixed point set B with components $B_1, \ldots, B_m$. The components of the fixed point set are complex submanifolds of M. For $\lambda \in \mathbb{C}^*$ and $p \in M$ let $\lambda \cdot p$ denote the action of λ on p. The $\mathbb{C}^*$-action extends to a meromorphic map

$$\mathbb{P}^1 \times \{p\} \longrightarrow M$$

hence $\lim_{\lambda\to 0} \lambda \cdot p$ and $\lim_{\lambda\to\infty} \lambda \cdot p$ exist in M. Clearly these limits are in B. There are two canonical decompositions of M into invariant complex submanifolds. Define

$$M_i^+ = \{p \in M | \lim_{\lambda\to 0} \lambda \cdot p \in B_i\}$$

for $i = 1, ..., m$. Each M_i^+ is a complex submanifold of M and

$$M = \bigcup M_i^+, \ \ 1 \leq i \leq m.$$

This is called the *plus decomposition* of M. Similarly the *minus decomposition* is defined as

$$M_i^- = \{p \in M | \lim_{\lambda \to \infty} \lambda \cdot p \in B_i\}$$

for $i = 1, ..., m$. Each M_i^- is a complex submanifold and similarly

$$M = \bigcup M_i^-, \;\; 1 \leq i \leq m.$$

There are two distinguished components of the fixed point set B, say B_1 and B_m, which are determined by the property that M_1^+ and M_m^- are open and dense in M. B_1 is called the *source* and B_m is called the *sink*.

2 $\mathbb{C}^*$-action on $G(k, n)$.

In this section we describe a particular $\mathbb{C}^*$-action on $G(k, n)$, the Grassmann manifold of k-planes in n-space. Fix a coordinate system on $\mathbb{C}^n$. We will use the representation of $G(k, n)$ by matrices. Any point $p \in G(k, n)$ can be represented by a $k \times n$-matrix A of rank k. Two such matrices A and B represent the same point in $G(k, n)$ if there is an invertible $k \times k$-matrix $g \in G(k, \mathbb{C})$ such that $gA = B$. For a $k \times n$-matrix A of rank k set $[A]=$ the row space of A.

Given a $k \times n$-matrix $A = (a_{ij}),\ 1 \leq i \leq k,\ 1 \leq j \leq n$ define two submatrices

$$A_1 = (a_{ij}), \;\; 1 \leq i, j \leq k$$

and

$$A_2 = (a_{ij}),\ 1 \leq i \leq k, \;\; k+1 \leq j \leq n.$$

A_1 is a $k \times k$-matrix and A_2 is a $k \times (n-k)$-matrix and $A = (A_1, A_2)$ is a partition of A.

Define a $\mathbb{C}^*$-action on $G(k, n)$

$$\mathbb{C}^* \times G(k, n) \longrightarrow G(k, n)$$

by

$$\lambda \cdot [A] = [(A_1, \lambda A_2)].$$

To describe the behaviour of this action define a subset X_{ij} of $G(k, n)$ as the set of all p in $G(k, n)$ which can be represented by a $k \times n$-matrix $A = (A_1, A_2)$ such that $\mathrm{rank} A_1 = i$ and $\mathrm{rank} A_2 = j$, where $k - \min\{k, n-k\} \leq i \leq k$ and

$0 \le j \le \min\{k, n-k\}$. Let $B = (B_1, B_2)$ be another $k \times n$-matrix representing p. Then there is an invertible $k \times k$-matrix g such that $gA = B$.

$$gA_1 = B_1 \quad \text{and} \quad gA_2 = B_2.$$

Hence $\text{rank}B_1 = \text{rank}(gA_1) = \text{rank}A_1 = i$ and similarly $\text{rank}B_2 = j$, and the following definition of X_{ij} is well defines:

$$X_{ij} = \left\{ [A] \in G(k,n) \;\middle|\; \text{rank}A_1 = i, \;\; \text{rank}A_2 = j \right\}$$

where $k - \min\{k, n-k\} \le i \le k$ and $0 \le j \le \min\{k, n-k\}$. Necessarily we have $i + j \ge k$; to see this, recall that A represents a point in $G(k,n)$ hence has rank k, and if A_1 has rank i then A_2 must supply at least the remaining $k - i$ ranks.

To describe the behaviour of the $\mathbb{C}^*$-action that is defined above we prove the following lemmas.

Lemma 1. *$X_{i\,k-i}$ are the fixed point components of the $\mathbb{C}^*$-action, $k - \min\{k, n-k\} \le i \le k$.*

Proof: Let $[A] \in X_{i\,k-i}$, $A = (A_1, A_2)$. We first show that $\lambda \cdot [A] = [A]$. If $i = 0$, then $A_1 = 0$, and if $i = k$, then $A_2 = 0$. In both cases $\lambda \cdot [A] = [A]$. Assume $0 < i < k$. Then there exists an invertible $k \times k$-matrix g such that gA is of the form

$$gA \;\; = \;\; \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

where $B_1 \in GL(i, \mathbb{C})$ and $B_2 \in GL(k-i, \mathbb{C})$. For $\lambda \in \mathbb{C}^*$ define h_λ to be the diagonal matrix $[1, \ldots, 1, 1/\lambda, \ldots, 1/\lambda]$, where the number of $1/\lambda$'s is $k - i$. We then have the following sequences of equalities:

$$\begin{aligned}
\lambda \cdot [A] &= \lambda \cdot [gA] \\
&= \lambda \cdot \left[\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \right] \\
&= \left[\begin{pmatrix} B_1 & 0 \\ 0 & \lambda B_2 \end{pmatrix} \right] \\
&= \left[h_\lambda \begin{pmatrix} B_1 & 0 \\ 0 & \lambda B_2 \end{pmatrix} \right] \\
&= \left[\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \right] \\
&= [A]
\end{aligned}$$

Thus we have proven that X_{ik-i} is a subset of the fixed point set. That in fact there are no other fixed points than $\cup X_{ik-i}$, $k - \min\{k, n-k\} \le i \le k$ follows from the results of the following two lemmas.

Lemma 2. *If* $[A] \in X_{ij}$*, then* $\lim_{\lambda\to 0} \lambda \cdot [A] \in X_{ik-i}$*, where*
$k - \min\{k, n-k\} \leq i \leq k,\ \ 0 \leq j \leq \min\{k, n-k\}\ \ i + j \geq k.$
In particular X_{m0} *is the source where* $m = k - \min\{k, n-k\}$*.*

Proof: If $i = 0$ or $i = k$, then X_{ij} is a component of the fixed point set as in Lemma 1. Assume $0 < i < k$. then there exists $g \in GL(k, \mathbb{C})$ such that

$$gA = \begin{pmatrix} & \vdots & 0 \\ B_1 & \vdots & \\ & \vdots & B_2 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & B_3 \end{pmatrix}$$

where $B_1 \in GL(i, \mathbb{C})$, $B_3 \in GL(k-i, \mathbb{C})$ and B_2 is a $(i+j-k) \times (n-k)$-matrix. Let h_λ be as in Lemma 1. then

$$h_\lambda \lambda g A = \begin{pmatrix} & \vdots & 0 \\ B_1 & \vdots & \\ & \vdots & \lambda B_2 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & B_3 \end{pmatrix}$$

and since $\lim_{\lambda\to 0} \lambda B_2 = 0$ we have

$$\begin{aligned} \lim_{\lambda\to 0} \lambda \cdot [A] &= \lim_{\lambda\to 0} [h_\lambda \lambda g A] \\ &= \left[\begin{pmatrix} B_1 & 0 \\ 0 & B_3 \end{pmatrix} \right]. \end{aligned}$$

This last matrix is clearly in X_{ik-i} as claimed. **QED**

Lemma 3. *If* $[A] \in X_{ij}$*, then* $\lim_{\lambda\to \infty} \lambda \cdot [A] \in X_{k-jj}$*, where*
$k - \min\{k, n-k\} \leq i \leq k,\ \ 0 \leq j \leq \min\{k, n-k\}.$
In Particular X_{k-mm} *is the sink, where* $m = \min\{k, n-k\}$*.*

Proof: If $i = 0$ or $i = k$, then X_{ij} is a fixed point component. Assume

$0 < i < k$. there exists $g \in GL(k, \mathbb{C})$ such that

$$gA = \begin{pmatrix} B_1 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ B_2 & \vdots & \\ & \vdots & B_3 \\ 0 & \vdots & \end{pmatrix}$$

where $B_1 \in GL(k-j, \mathbb{C})$, $B_3 \in GL(j, \mathbb{C})$ and B_2 is a $(i+j-k) \times k$-matrix. Then

$$\begin{aligned} \lim_{\lambda \to \infty} \lambda \cdot [A] &= \lim_{\lambda \to \infty} [\lambda h_\lambda g A] \\ &= \lim_{\lambda \to \infty} \left[\begin{pmatrix} B_1 & 0 \\ \lambda^{-1} B_2 & \\ & B_3 \\ 0 & \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} B_1 & 0 \\ 0 & B_3 \end{pmatrix} \right]. \end{aligned}$$

This last matrix is in X_{k-jj} as desired. **QED**

These last two lemmas show that X_{ik-i} for $k - \min\{k, n-k\} \leq i \leq k$ are the only fixed point components and thus complete the proof of lemma 1.

We can apply these lemmas to examine the behaviour of Schubert cells under the action of $\mathbb{C}^*$ on the Grassmann manifold. we will adopt the terminology of Griffiths and Harris on Schubert cells. For details refer to [GH, pp. 195-196].

Let $\{e_1, ..., e_n\}$ be the standard basis for $\mathbb{C}^n$ and $V_i = \text{span}\{e_1, ..., e_i\}$. Then $\{V_1, ..., V_n\}$ defines a flag. For any nonincreasing sequence of nonnegative integers between 0 and $n-k$ define a cell

$$W_a = \left\{ [\Lambda] \in G(k, n) \;\middle|\; \dim(\Lambda \cap V_{n-k+i-a_i}) = i \right\}.$$

The sequence of nonincreasing integers $a = (a_1, ..., a_k)$ with $0 \leq a_i \leq n-k$ is called a Schubert symbol. For $[\Lambda] \in G(k, n)$, let A be a $k \times n$-matrix such that $[A] = [\Lambda]$. If $[A] \in W_a$ for some Schubert symbol $a = (a_1, ..., a_k)$, then the rank of the first $k \times (n-k+i-a_i)$ minor is i and the rank of the last $k \times (k-i+a_i)$ minor is $k-i$. The closure of W_a

$$\overline{W_a} = \left\{ [\Lambda] \in G(k, n) \;\middle|\; \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i \right\}$$

is called a Schubert variety. If A is a matrix representing $[\Lambda]$ as above, then $[\Lambda]$ is in $\overline{W_a}$ iff the rank of the first $k \times (n-k+i-a_i)$ minor of A is at least i and the rank of the last $k\times(k-i+a_i)$ minor of A is at most $k-i$. It is well known that $\overline{W_a}$ is an analytic subvariety of $G(k,n)$ and the homology class of $\overline{W_a}$, denoted by σ_a, is independent of the flag used in its definition, [GH, p. 196]. σ_a is called the Schubert cycle corresponding to $a=(a_1,...,a_k)$. Regarding the behaviour of Schubert cycles under the $\mathbb{C}^*$-action we give the following corollary to the above lemmas:

Corollary 1. *All Schubert cycles of positive codimension in $G(k,2k)$ lie in X_{ij}'s where $j<k$. In particular they do not flow to the sink, i.e. if $p\in\overline{W_a}$ then $\lim_{\lambda\to\infty}\lambda\cdot p$ is not in the sink.*

Proof: The codimension of $\overline{W_a}$ for $a=(a_1,...,a_k)$ is $\sum a_i$, [GH, p. 196]. It suffices to prove the corollary for $a=(1,0,...,0)$. For $[\Lambda]\in W_a$ let $A=(A_1,A_2)$ be a matrix representation where A is a $k\times n$-matrix of rank k, and A_1,A_2 are $k\times k$-matrices. The rank of the last $k\times k$ minor of A is at most $k-1$. Hence in particular the rank of A_2 is not k, therefore $[A]$ is not in X_{ik}. Since the only points that flow to the sink belong to the components of the form X_{ik}, $[\Lambda]$ does not flow to the sink. In general if $a=(a_1,...,a_k)$ with $a_1\geq 1$ then the last $k\times(k+a_1-1)$ minor has rank at most $k-1$. Since $k+a_1-1\geq k$, the rank of A_2 cannot be k. Hence $\overline{W_a}$ does not flow to the sink. If $a_1=0$, then $a=(0,...,0)$ and $\overline{W_a}$ does not have positive codimension. **QED**

Using the same notation as in the previous corollary we can generalize as follows:

Corollary 2. *Let $\overline{W_a}$, $a=(a_1,...,a_k)$, be a Schubert variety in $G(k,n)$, where $a_1\geq n-2k+1$. Then $\overline{W_a}$ does not flow to the sink if $n\geq 2k$.*

Proof: Let $A=(A_1,A_2)$ be a $k\times n$-matrix with rank k representing a point $[A]$ in $\overline{W_a}$. A_1 is a $k\times(n-k)$-matrix and $[A]$ will flow to the sink if rank A_2 is maximal. Since $n\geq 2k$ means $n-k\geq k$, the maximal rank of A_2 is k. the rank of the last $k\times(k+a_1-1)$ minor of A is at most $k-1$. By assumption $k+a_1-1\geq n-k$, therefore the rank of A_2 cannot be k. Hence $\overline{W_a}$ does not flow to the sink. **QED**

3 Examples

In examples 1 and 2 we assume that the $\mathbb{C}^*$-action of the previous section is defined on the spaces $G(2,4)$ and $G(4,9)$.

1) G(2, 4). In $G(2,4)$ we have defined the following sets:

$$X_{20}, X_{21}, X_{22}, X_{11}, X_{12}, X_{02}.$$

The first three sets are the fixed point sets. As $\lambda \to 0$ the elements of X_{21} and X_{22} flow to the source X_{20}, and the elements of X_{12} flow to X_{11}. As $\lambda \to \infty$ the elements of X_{22} and X_{12} flow to the sink X_{02}, and the elements of X_{21} flow to X_{11}.

See Figure 1 for the direction of these flows for each X_{ij} as $\lambda \to \infty$.

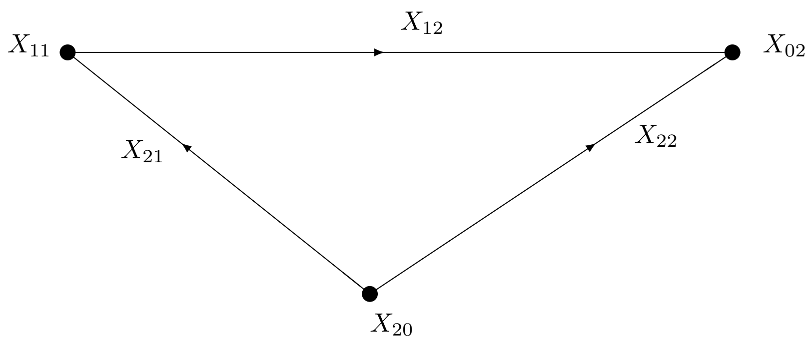

Fig. 1

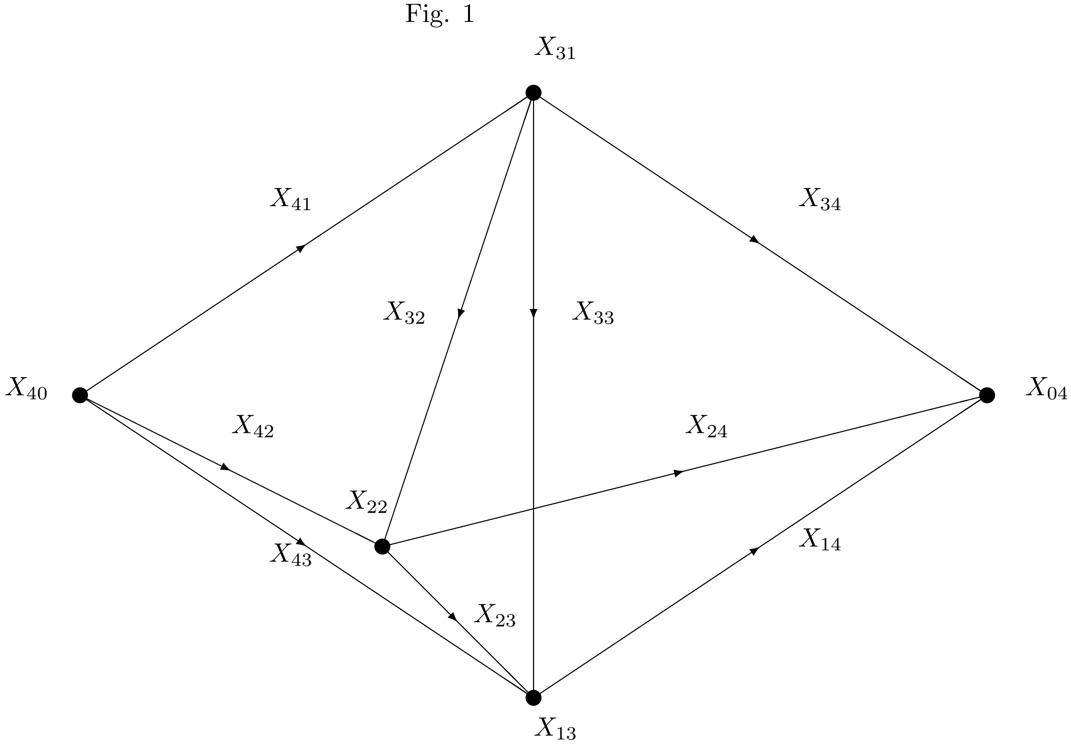

Fig. 2

2) G(4, 9). For the direction of flow as $\lambda \to \infty$ see Figure 2. From the decomposition of $G(4,9)$ into X_{ij} it can be seen that the points that lie in

$$X_{13} \cup X_{33} \cup X_{31} \cup X_{32} \cup X_{22} \cup X_{23}$$

do not flow to the sink or the source under the action of $\mathbb{C}^*$.

4 $\mathbb{C}^*$-actions on Grassmann Bundles

This section defines in the compact Kaehler case the Grassmann Graph construction of [BFM, pp. 120-121].

Let E, F be vector bundles of ranks k and n, respectively, on an analytic space M. Let $G(k, \mathrm{E} \oplus \mathrm{F}) \to M$ denote the Grassmann bundle whose fiber at each $x \in M$ is $G(k, \mathrm{E}_x \oplus \mathrm{F}_x)$, the Grassmannian of k-planes in $\mathrm{E}_x \oplus \mathrm{F}_x$. Define a $\mathbb{C}^*$-action on $G(k, \mathrm{E} \oplus \mathrm{F})$ as the fibrewise $\mathbb{C}^*$-action. Let

$$\pi_1 : \mathrm{E} \oplus \mathrm{F} \longrightarrow \mathrm{E}$$
$$\pi_2 : \mathrm{E} \oplus \mathrm{F} \longrightarrow \mathrm{F}$$

and

$$\pi : G(k, \mathrm{E} \oplus \mathrm{F}) \longrightarrow M$$

be the projections. Any $p \in G(k, \mathrm{E} \oplus \mathrm{F})$ is represented by a k-plane H in $\mathrm{E}_x \oplus \mathrm{F}_x$ where $x = \pi(x)$. $\pi_1(H)$ and $\pi_2(H)$ are linear subspaces of E_x and F_x, respectively. The total space $G(k, \mathrm{E} \oplus \mathrm{F})$ can be decomposed into $\mathbb{C}^*$-equivariant subbundles

$$X_{ij} = \big\{ [H] \in G(k, \mathrm{E} \oplus \mathrm{F}) \;\big|\; \dim \pi_1(H) = i,\ \dim \pi_2(H) = j \;\big\}$$

where $k - \min(k,n) \leq i \leq k$, $\;0 \leq j \leq \min(k,n)$, and $i + j \geq k$. It is easy to see that

$$X_{ij} \cong G(i, \mathrm{E}) \times G(j, \mathrm{F}) \;\text{ if }\; i + j = k,$$

which are the fixed point sets of the $\mathbb{C}^*$-action. Let

$$\mathrm{Hom}(\mathrm{E}, \mathrm{F}) \longrightarrow M$$

be the bundle of morphisms from E to F and let

$$j : \mathrm{Hom}(\mathrm{E}, \mathrm{F}) \longrightarrow G(k, \mathrm{E} \oplus \mathrm{F})$$

be the natural inclusion defined fibrewise as

$$j_x(\Phi) = \operatorname{graph}(\Phi \mid \mathrm{E}_x) = \big\{\big(e, \Phi(e)\big) \in \mathrm{E}_x \oplus \mathrm{F}_x\big\}.$$

Recall that $\mathbb{C}$ can be imbedded into $\mathbb{P}^1$ as

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{P}^1 \\ \lambda &\longrightarrow [1 : \lambda], \end{aligned}$$

[BFM, p. 120]. Define a $\mathbb{C}^*$-action on $G(k, \mathrm{E} \oplus \mathrm{F}) \times \mathbb{P}^1$

$$\mathbb{C}^* \times G(k, \mathrm{E} \oplus \mathrm{F}) \times \mathbb{P}^1 \longrightarrow G(k, \mathrm{E} \oplus \mathrm{F}) \times \mathbb{P}^1$$

as

$$\big(\lambda, p, [\lambda_0 : \lambda_1]\big) \longrightarrow \big(\lambda \cdot p, [\lambda_0 : \lambda\lambda_1]\big)$$

where $\lambda \cdot p$ is the $\mathbb{C}^*$-action which is defined above. Also define the $\mathbb{C}^*$-action on $M \times \mathbb{C}$,

$$\mathbb{C}^* \times M \times \mathbb{C} \longrightarrow M \times \mathbb{C}$$

as

$$\big(\lambda, x, t\big) \longrightarrow \big(x, \lambda t\big).$$

Every $\Phi \in \operatorname{Hom}(\mathrm{E}, \mathrm{F})$ defines an equivariant imbedding $s(\Phi)$ of $M \times \mathbb{C}$ into $G(k, \mathrm{E} \oplus \mathrm{F}) \times \mathbb{P}^1$,

$$s(\Phi) : M \times \mathbb{C} \longrightarrow G(k, \mathrm{E} \oplus \mathrm{F}) \times \mathbb{P}^1$$

where

$$s(\Phi)(x, \lambda) = \big(\big[j_x(\lambda\Phi_x)\big], [1 : \lambda]\big).$$

$s(\Phi)(M, \lambda)$ is the graph of $\lambda\Phi$. Now define

$$\mathbf{Z}_\infty = \lim_{\lambda \to \infty} s(\Phi)(M, \lambda).$$

Theorem 1. *If M is a compact Kaehler manifold, then for any $\Phi \in \operatorname{Hom}(\mathrm{E}, \mathrm{F})$ the corresponding $\mathbf{Z}_\infty$ is an analytic cycle.*

Proof: Let $\rho : \mathbb{C}^* \times G(k, \mathrm{E} \oplus \mathrm{F}) \to G(k, \mathrm{E} \oplus \mathrm{F})$ be the $\mathbb{C}^*$-action defined above. Consider M as a subspace of $G(k, \mathrm{E} \oplus \mathrm{F})$ by the imbedding $s(\Phi)(M, \lambda)$; i.e. identify M and the graph of Φ. define a holomorphic map

$$A : M \times \mathbb{C}^* \longrightarrow G(k, \mathrm{E} \oplus \mathrm{F})$$

as

$$A(m,t) = s(\Phi)(m,t),$$

where $m \in M$ and $t \in \mathbb{C}^*$. This map is equivariant with respect to ρ and the trivial action of $\mathbb{C}^*$ on $M \times \mathbb{C}^*$, multiplication in the second component; for if $\lambda \in \mathbb{C}^*$ then

$$\begin{aligned} A(m, \lambda \cdot t) &= s(\Phi)(m, \lambda t) \\ &= s(\lambda\Phi)(m,t) \\ &= \lambda \cdot s(\Phi)(m,t) \\ &= \rho(\lambda, s(\Phi)(m,t)) \\ &= \rho(\lambda, A(m,t)) \end{aligned}$$

hence equivariance. But Sommese has shown that if $\psi : Y \times \mathbb{C}^* \to X$ is a holomorphic map equivariant with respect to the trivial action of $\mathbb{C}^*$ on $Y \times \mathbb{C}^*$ and the action of $\mathbb{C}^*$ on X with fixed points then ψ extends meromorphically to $Y \times \mathbb{P}^1$, [So, p. 111 (Lemma II-B)]. Thus A extends meromorphically to

$$A' : M \times \mathbb{P}^1 \longrightarrow G(k, \mathrm{E} \oplus \mathrm{F}).$$

Let T be the closure of the graph of A in $M \times \mathbb{P}^1 \times G(k, \mathrm{E} \oplus \mathrm{F})$.

By the definition of a meromorphic map, T is an analytic space. Since

$$\mathbf{Z}_\infty = T \cap \big(M \times \{\infty\} \times G(k, \mathrm{E} \oplus \mathrm{F})\big),$$

the intersection of two analytic spaces, then $\mathbf{Z}_\infty$ is analytic as desired. **QED**

$\mathbf{Z}_\infty$ is called the cycle at infinity corresponding to the map Φ. Notice that there is an alternate definition of $\mathbf{Z}_\infty$ see [BFM, p. 121]; define an imbedding of $M \times \mathbb{P}^1$ into $M \times \mathbb{P}^1 \times G(k,n)$

$$i : M \times \mathbb{P}^1 \longrightarrow M \times \mathbb{P}^1 \times G(k,n)$$

as

$$i(m, [\lambda_0 : \lambda_1]) = [\{(m, [\lambda_0 : \lambda_1], (e,f)) \in \mathrm{E}_m \oplus \Phi_m(\mathrm{E}_m) | \lambda_0 f = \lambda_1 \Phi(e)\}]$$

Let W be the closure of $i(M \times \mathbb{P}^1)$ in $M \times \mathbb{P}^1 \times G(k,n)$.

$$W = \overline{i(M \times \mathbb{P}^1)}$$

Then

$$\mathbf{Z}_\infty = W \cup \big(M \times \{\infty\} \times G(k, \mathrm{E} \oplus \mathrm{F})\big).$$

In the algebraic category W is an algebraic variety but in the analytic category the observation that W can be obtained through a $\mathbb{C}^*$-action with fixed points on a compact Kaehler manifold is crucial in concluding that it is analytic.

Clearly $\{Z_\lambda = s(\Phi)(M,\lambda)\}$ defines a family of cycles which are algebraically and hence homologically equivalent.

5 Graphs of Complexes

In this section we define the Grassmann Graph construction and the cycle at infinity associated to a complex of vector bundles. This construction was first introduced by MacPherson and used by Baum, Fulton and MacPherson to prove Riemann-Roch theorem for singular algebraic varieties, [BFM] and [Mc].

Consider a complex of vector bundles on M,

$$(\mathrm{E.}): \quad 0 \longrightarrow \mathrm{E}_m \longrightarrow \mathrm{E}_{m-1} \longrightarrow \cdots \longrightarrow \mathrm{E}_0 \longrightarrow 0.$$

Denote the maps by γ_i, i.e.

$$\gamma_i : \mathrm{E}_i \longrightarrow \mathrm{E}_{i-1}$$

where $i = 0, ..., m$.

Assume that there is a subvariety S of M such that (E.) is exact on $M - S$.

Let

$$G_i = G(\mathrm{rank}\mathrm{E}_i,\ \mathrm{E}_i \oplus \mathrm{E}_{i-1}\ ), \quad i = 1, ..., m.$$

and let

$$\tau_i \longrightarrow G_i \quad \text{the tautological bundle,} \quad i = 1, ..., m.$$

Define

$$G = G_0 \times_M \cdots \times_M G_m$$

where $\times_M$ denotes the bundle product on M. On G let τ_i denote the pull back of $\tau_i \to G_i$ by the projection $\mathrm{pr}_i : G \to G_i$ of the i-th component, $i = 0, ..., m$.

Let

$$\tau = \tau_0 - \tau_1 + \cdots + (-1)^m \tau_m$$

be the virtual tautological bundle on G. Recalling the definition of s from the previous section, for any $\lambda \in \mathbb{C}$ define an imbedding

$$s^i_\lambda : M \longrightarrow G_i$$

as

$$s^i_\lambda(x) = s(\gamma_i)(x, \lambda)$$

where $i = 0, ..., m$. Then define for any $\lambda \in \mathbb{C}$ an imbedding

$$s_\lambda : M \longrightarrow G$$

by

$$s_\lambda(x) = \big(s^0_\lambda(x), ..., s^m_\lambda(x)\big).$$

Let $\mathbf{Z}_\infty$ again denote $s_\lambda(M)$ for $\lambda \in \mathbb{C}$. Then define

$$\mathbf{Z}_\infty = \lim_{\lambda \to \infty} Z_\lambda$$

to be the cycle at infinity corresponding to the complex (E.).

Let $\pi : G \to M$ be the natural projection. Recalling that S is the set off which (E.) is exact we have the following result: (For proofs see [BFM, p. 121].)

Theorem (Baum, Fulton, MacPherson) *The cycle $\mathbf{Z}_\infty$ has a unique decomposition $\mathbf{Z}_\infty = \mathbf{Z}_* + \mathbf{M}_*$, where*
1) π maps M meromorphically onto M.
2) $\pi : \mathbf{M}_* - \pi^{-1}(S) \longrightarrow M - S$ is a biholomorphism.
3) π maps Z into S.
4) τ restricts on $\mathbf{M}_*$ to the zero bundle.

Remark. By Theorem 1 of the previous section, $\mathbf{Z}_\infty$ is a product of analytic cycles in the product bundle G, hence this theorem can be stated in the analytic category as above. Any cycle can be written as a sum of irreducible cycles. the decomposition of $\mathbf{Z}_\infty$ is such a sum. For a proof of (4) see [BFM, p. 122].

Finally we define two residues on S. Let E be the virtual bundle $\mathrm{E}_0 - \mathrm{E}_1 + \cdots + (-1)^m \mathrm{E}_m$ on M. Then $\tau \mid Z_0$ is isomorphic to E since $Z_0 \cong M$. Since Z_0 and $\mathbf{Z}_\infty$ are rationally equivalent

$$c(\mathrm{E}) \cap [M] = c(\tau) \cap Z_0 = c(\tau) \cap \mathbf{Z}_\infty$$

where $c(\cdot)$ denotes the Chern class and $\cap$ denotes the cap product. Since $\mathbf{Z}_\infty$ decomposes

$$\begin{aligned} c_i(\tau) \cap \mathbf{Z}_\infty &= c_i(\tau) \cap (\mathbf{Z}_* + \mathbf{M}_*) \\ &= c_i(\tau) \cap \mathbf{Z}_* + c_i(\tau) \cap \mathbf{M}_* \\ &= c_i(\tau) \cap \mathbf{Z}_* \end{aligned}$$

where $i > 0$ and the last equality follows since $\tau \mid \mathbf{M}_* = 0$ by (4) of the above theorem.

Define

$$c_S^i(\mathrm{E.}) = \pi_*\big(c_i(\tau) \cap \mathbf{Z}_*\big) \in H_*(S : \mathbb{C}).$$

Similarly let $ch(\cdot)$ denote the Chern character, then

$$\begin{aligned} ch(\mathrm{E}) \cap [M] &= ch(\tau) \cap Z_0 \\ &= ch(\tau) \cap \mathbf{Z}_\infty \\ &= ch(\tau) \cap \mathbf{Z}_* + ch(\tau) \cap \mathbf{M}_* \\ &= ch(\tau) \cap \mathbf{Z}_*. \end{aligned}$$

Similarly define

$$ch_S(\mathrm{E.}) = \pi_*\big(ch(\tau) \cap \mathbf{Z}_*\big) \in H_*(S; \mathbb{C}).$$

For basic properties of $ch(\mathrm{E.})$ in the algebraic category see [BFM, pp. 121-126]. We will use $c_S^i(\mathrm{E})$ for calculating the Baum-Bott residue of singular holomorphic foliations in the next chapter.

6 Examples

1) Let M be a compact Kaehler manifold of dimension n with tangent bundle T. Let L be a line bundle on M and let $\alpha \in \mathrm{Hom}(\mathrm{L}^*, \mathrm{T})$ with isolated zeros Z. α is called a *meromorphic vector field*. Let $p \in Z$. Choose an open neighbourhood U of p such that (i) there are coordinate functions $z_1, ..., z_n$ on U and (ii) there is a global generator l^* of L on U and (iii) $U \cap Z = \{p\}$. Then $\alpha(l^*)$ is a holomorphic vector field on U given as

$$\alpha(l^*) \mid x = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial z_i}|x, \quad x \in U$$

where $a_i(\cdot)$ are holomorphic functions on U. Any element of $\mathrm{L}^*|x$ is of the form $c \cdot (l^*|x)$ for some $c \in \mathbb{C}$, and

$$\alpha(cl^*)|x = \sum_{i=1}^{n} ca_i(x) \frac{\partial}{\partial z_i}|x, \quad x \in U.$$

This defines a point in $\mathrm{L}^* \oplus \mathrm{T}|x$,

$$(cl^*, \alpha(cl^*)|x) = (c, ca_1(x), ..., ca_n(x)) \in \mathrm{L}^* \oplus \mathrm{T}|x, \quad x \in U.$$

Hence for $x \in U - p$, the graph Γ of α in $U \times \mathbb{P}(\mathrm{L}^* \oplus \mathrm{T}) \cong U \times \mathbb{P}^n$ is given as

$$\Gamma = \{(x, [1 : a_1(x) : \cdots : a_n(x)]) \in U \times \mathbb{P}^n.$$

Define a $\mathbb{C}^*$-action on $U \times \mathbb{P}(\mathrm{L}^* \oplus \mathrm{T}) \cong U \times \mathbb{P}^n$

$$\mathbb{C}^* \times U \times \mathbb{P}^n \longrightarrow U \times \mathbb{P}^n$$

as

$$(\lambda, x, [y_0 : \cdots : y_n]) = (x, [y_0 : \lambda y_1 : \cdots : \lambda y_n]).$$

Consider $\lim_{\lambda\to\infty} \lambda \cdot \Gamma = \mathbf{Z}_\infty$.

$$\lambda \cdot \Gamma = \{(x, [1 : \lambda a_1(x) : \cdots : \lambda a_n(x)]) \in U \times \mathbb{P}^n\}.$$

If $x \in U - p$, then

$$\lim_{\lambda\to\infty} \lambda \cdot \Gamma = \{(x, [0 : a_1(x) : \cdots : a_n(x)]) \in U \times \mathbb{P}^n\}$$

and hence $\mathbf{Z}_\infty | U - p \cong U - p$. To find $\mathbf{Z}_\infty | p$, define a holomorphic function

$$F : U \longrightarrow \mathbb{C}^n$$

as

$$F(x) = (a_1(x), ..., a_n(x)).$$

Then $F(p) = 0$ and $F(U)$ is an open neighbourhood of the origin. For any point $[c_1 : \cdots : c_n] \in \mathbb{P}^{n-1}$, let D be the line in $\mathbb{C}^n$ that passes through $(c_1, ..., c_n)$ and the origin. Consider the set

$$C = \{x \in U \mid F(x) \in U \cap D\}.$$

Then C is a union of holomorphic curves $\{\zeta_1, ..., \zeta_k\}$ passing through p. We may assume without loss of generality that these curves do not intersect in $U - p$. The number k will be referred to as the degree of F at p. Let ζ be one of these curves. Let $\{p_m\} \in \zeta$ be a sequence of points such that

$$\lim_{m\to\infty} p_m = p.$$

The graph of Γ on any one of these p_m's can be written as

$$\begin{aligned}\Gamma | p_m &= \{(p_m, [1 : a_1(p_m) : \cdots : a_n(p_m)]) \in U \times \mathbb{P}^n\} \\ &= \{(p_m, [1 : c_1 : \cdots : c_n]) \in U \times \mathbb{P}^n\}\end{aligned}$$

since $F(p_m) \in D$. Then

$$\begin{aligned}\lim_{m\to\infty} \lim_{\lambda\to\infty} \lambda \cdot \Gamma \bigm| p_m &= \lim_{m\to\infty} \lim_{\lambda\to\infty} \{(p_m, [1 : \lambda c_1 : \cdots : \lambda c_n]) \in U \times \mathbb{P}^n\} \\ &= \lim_{m\to\infty} \{(p_m, [0 : c_1 : \cdots : c_n]) \in U \times \mathbb{P}^n\} \\ &= \{(p, [0 : c_1 : \cdots : c_n]) \in U \times \mathbb{P}^n\}.\end{aligned}$$

On the other hand

$$\Gamma \bigm| p = \{(p, [1 : 0 : \cdots : 0])\}$$

and

$$\lambda \cdot \Gamma \bigm| p = \Gamma \bigm| p.$$

Therefore

$$\mathbf{Z}_\infty \bigm| p \cong k \cdot \mathbb{P}^{n-1} \ + \ \{(p, [1 : 0 : \cdots : 0])\}$$

as cycles. $\mathbf{Z}_\infty$ can be decomposed uniquely into two cycles

$$\mathbf{Z}_\infty = U_* + Z_*,$$

where U_* is bimeromorphic to U and Z_* lies over p, i.e. if

$$\pi : \mathbf{Z}_\infty \longrightarrow U$$

is the usual projection induced by the natural projection

$$U \times \mathbb{P}^n \longrightarrow U,$$

then $\pi(Z_*) = p$. Hence $Z_* \cong (k-1)\mathbb{P}^{n-1} + \{\text{point}\}$ as cycles.

Let τ' be the tautological line bundle on $\mathbb{P}^n$ and let τ denote the pullback bundle on $U \times \mathbb{P}^n$. Let w be a dual hyperplane class in $H^*(\mathbb{P}^{n-1}; \mathbb{C})$.

$$c(\tau | U_*) = 1 - w$$

since τ restricts to the tautological bundle on $\mathbb{P}^{n-1}$, where $c(\cdot)$ is the total Chern class. Let $ch(\cdot)$ denote the Chern character. Then

$$\begin{aligned} ch(\tau) \cap Z_* &= (e^{-w} \cap [(k-1)\mathbb{P}^{n-1} + \{\text{point}\}]) \\ &= (e^{-w} \cap [(k-1)\mathbb{P}^{n-1}]) + (e^{-w} \cap \{\text{point}\}]) \\ &= \big((\sum_{i=0}^{n-1} \frac{(-1)^i}{i!} w^i) \cap [(k-1)\mathbb{P}^{n-1}] \big) + 1 \\ &= \big(\frac{(k-1)(-1)^{n-1}}{(n-1)!} + 1 \big) + \frac{(k-1)(-1)^{n-2}}{(n-2)!} w^{n-2} \cap [\mathbb{P}^{n-1} + \cdots \\ &+ (k-1)[\mathbb{P}^{n-1}]. \end{aligned}$$

Let

$$\pi_* : H_*(Z_*; \mathbb{C}) \longrightarrow H_*(p; \mathbb{C})$$

be the map induced by π on the homology classes. Notice that $H_*(\{p\}; \mathbb{C}) \cong \mathbb{C}$ hence π_* is zero on positive dimensional cycles and maps only the 0-cycles.

Define a local residue

$$ch_p(\alpha) = \pi_*\big(ch(\tau) \cap Z_*\big).$$

Then

$$ch_p(\alpha) = 1 + \frac{(k-1)(-1)^{n-1}}{(n-1)!} \tag{1}$$

Define the total residue as:

$$ch(\alpha) = \sum_{p \in Z} ch_p(\alpha) \tag{2}$$

It follows that

$$\begin{aligned} ch(\alpha) &= \sum_{p \in Z}\big(1 + \frac{(k(p)-1)(-1)^{n-1}}{(n-1)!}\big) \\ &= \#Z + \frac{(-1)^{n-1}}{(n-1)!}\sum_{p \in Z}(k(p)-1) \\ &= \#Z + \frac{(-1)^{n-1}}{(n-1)!}\big(\sum_{p \in Z} k(p) - \sum_{p \in Z} 1\big) \\ &= \#Z - \frac{(-1)^{n-1}}{(n-1)!}\#Z + \frac{(-1)^{n-1}}{(n-1)!}\sum_{p \in Z} k(p) \\ &= \big[\frac{(n-1)! - (-1)^{n-1}}{(n-1)!}\big]\#Z + \frac{(-1)^{n-1}}{(n-1)!}\sum_{p \in Z} k(p) \end{aligned}$$

where $k(p)$ is the degree of F at p as described above and $\#Z$ is cardinality of Z, without counting multiplicity. But it is shown in [GH, page 663-666] that

$$\sum_{p \in Z} k(p) = \sum_{p \in Z} \mathrm{res}\big\{\frac{\det A}{a_1 \cdots a_n}\big\}$$

where $A = (\partial a_i / \partial z_j)$ and $\mathrm{res}\{\cdot\}$ is the Grothendieck residue symbol. It is also known that

$$\sum_{p \in Z} \mathrm{res}\big\{\frac{\det A}{a_1 \cdots a_n}\big\} = c_n(\mathrm{T} - \mathrm{L}^*).$$

This is the meromorphic vector field theorem of Baum and Bott, [BB1]. Since $n = \dim M$, we have

$$c_n(\mathrm{T} - \mathrm{L}^*) = c_n(\mathrm{L} \otimes \mathrm{T}).$$

Hence

$$ch(\alpha) = \big[\frac{(n-1)! - (-1)^{n-1}}{(n-1)!}\big]\#Z + \frac{(-1)^{n-1}}{(n-1)!} c_n(\mathrm{L} \otimes \mathrm{T}). \tag{3}$$

If L^* imbeds into T by α, then $Z_* = 0$ and $Z = \emptyset$. Hence $ch(\alpha) = 0$ and $\#Z = 0$. Then equation (3) reduces to

$$0 = c_n(\mathrm{L} \otimes \mathrm{T})$$

which exhibits Bott's vanishing theorem, [B2].

If L is trivial and α has only nondegenerate zeroes then $k(p) = 1$ for all $p \in Z$ and from equation (1) we find that

$$ch_p(\alpha) = 1 \quad \text{for all} \quad p \in Z.$$

Hence by equation (2)

$$ch(\alpha) = \#Z.$$

Then equation (3) becomes

$$\#Z = c_n(\mathrm{T})$$

which is a consequence of the Hopf formula.

2) Let E, F be vector bundles on M and $\psi \in \mathrm{Hom}(\mathrm{E}, \mathrm{F})$. Then the graph $\Gamma(\psi)$ of ψ gives rise to a cycle at infinity $\mathbf{Z}_\infty$. We let $B_0, ..., B_m$ be the components of the fixed point set B of $G(k, \mathrm{E} \oplus \mathrm{F})$. Then $(\mathbf{Z}_\infty \cap B)_p = (\mathbf{Z}_\infty \cap B_0)_p \cup (\mathbf{Z}_\infty \cap B_i)_p$ for $p \in M$, where $i = \mathrm{rank}\psi_p$. It is of course possible that $(B_0)_p$ is empty at that point. This is because $\Gamma(\psi)$ and $\mathbf{Z}_\infty$ intersect E in the same set, namely the kernel of ψ. If i is the largest integer for any $p \in M$ such that $(\mathbf{Z}_\infty \cap B)_p = (\mathbf{Z}_\infty \cap B_0)_p \cup (\mathbf{Z}_\infty \cap B_i)_p$, then we say that "$\psi$ intersects the fixed point set generically at i". Then the generic rank of ψ is i. In particular let K be the curvature of E, then $K \in \mathrm{Hom}(\mathrm{E}, \Lambda^2\mathrm{T} \otimes \mathrm{E})$ and we have its graph in $G(\mathrm{rank}\mathrm{E}, \mathrm{E} \oplus \lambda^2\mathrm{T}^* \otimes \mathrm{E})$. If K intersects the fixed point set generically at i then $c_j(\mathrm{E}) = 0$ for $j > i$. Conversely if $c_j(\mathrm{E}) = 0$ for $j > i$ for some i then K intersects the fixed point set generically at t for some $t \leq i$. This is because $(\mathbf{Z}_\infty \cap B)_p$ contains $(B_i)_p$ iff rank ψ_p is i and rank is lower semicontinuous.

3) We want to show that the Hironaka Blow-up at a point can be recovered as a Grassmann Graph construction. The problem is local so let M be an open set in $\mathbb{C}^n$. Define two trivial bundles L and F as

$$\mathrm{L} = M \times \mathbb{C} \ \text{ and } \ \mathrm{F} = M \times \mathbb{C}^n.$$

Define a morphism $\theta \in \mathrm{Hom}(\mathrm{L}, \mathrm{F})$ as:

$$\theta(p, t) = (p, tp) \ \text{ for } \ p \in \mathbb{C}^n, \ t \in \mathbb{C}.$$

The cycle at infinity $\mathbf{Z}_\infty$ corresponding to θ intersects the sink of $G(1, \mathrm{L} \oplus \mathrm{F})$ in $\mathbf{M}_*$, that is $\mathbf{Z}_\infty = \mathbf{M}_* + \mathbf{Z}_*$. $\mathbf{M}_*$ is the Hironaka Blow-up of M at the origin.

We can see this as follows. Let $p = (p_1, ..., p_n) \in M = \mathbb{C}^n$. We also identify $\mathbb{P}(\mathrm{L} \oplus \mathrm{F})$ with $\mathbb{P}^n$. There is a $\mathbb{C}^*$-action

$$\mathbb{C}^* \times M \times \mathbb{P}^n \longrightarrow M \times \mathbb{P}^n$$

given as

$$\big(\lambda, p, [y_0 : y_1 : \cdots : y_n]\big) \to \big(p, [y_0 : \lambda y_1 : \cdots : \lambda y_n]\big).$$

The graph of θ has the form

$$\Gamma(\theta) = \big\{(p, [1 : p_1 : \cdots : p_n]) \in M \times \mathbb{P}^n\big\}.$$

The $\mathbb{C}^*$-action moves $\Gamma(\theta)$ as

$$\lambda \cdot \Gamma(\theta) = \big\{(p, [1 : \lambda p_1 : \cdots : \lambda p_n]) \in M \times \mathbb{P}^n\big\}.$$

Consider the usual imbedding of $\mathbb{C}^*$ in $\mathbb{P}^1$ as $\lambda = [1 : \lambda] = [\lambda_0 : \lambda_1]$, where $\lambda = \lambda_1/\lambda_0$. Since $\lambda \to \infty$ iff $\lambda_0 \to 0$ with $\lambda_1 \neq 0$, we have the following limit

$$\begin{aligned}
\mathbf{Z}_\infty &= \lim_{\lambda \to \infty} \lambda \cdot \Gamma(\theta) \\
&= \lim_{\lambda_0 \to 0} \big\{(p, [\lambda_0 : \lambda_1 p_1 : \cdots : \lambda_1 p_n]) \in M \times \mathbb{P}^n\big\} \\
&= \big\{(p, [0 : \lambda_1 p_1 : \cdots : \lambda_1 p_n]) \in M \times \mathbb{P}^n\big\}.
\end{aligned}$$

Clearly $[0 : \lambda_1 p_1 : \cdots : \lambda_1 p_n]$ can be considered as a point $[x_1 : \cdots : x_n]$ in $\mathbb{P}^{n-1}$ such that

$$p_j x_i = p_i x_j, \;\; i \neq j, \;\; 1 \leq i, j \leq n.$$

From here it is easy to see that the intersection of $\mathbf{Z}_\infty$ with the sink of the $\mathbb{C}^*$-action is the Hironaka blow-up of M at the origin.

CHAPTER 3
SINGULAR HOLOMORPHIC FOLIATIONS

0 Introduction

Foliations arise naturally in mathematics, such as in submersions, group actions and differential equations. For an introduction to the subject we refer to the expository article of Lawson on foliations, [L]. Lawson claims:

> One of the reasons that foliations interest people in geometry is that they constitute a class of structures on manifolds which is complicated enough to shed light on the general situation but has certain geometric aspects that make it tractable, [L].

In this chapter we will investigate residue properties of singular holomorphic foliations. In Section 1 we summarize some of the basic ideas. In Section 2 we define Baum-Bott residues, see [BB2]. Section 3 gives Suwa's recent contribution, see [Su]. The main theme of residues is given in Section 4 where we calculate Baum-Bott residues using Nash Blow-up and Grassmann Graph construction.

1 Preliminaries

A holomorphic foliation L of rank k on a complex manifold M of dimension n is a decomposition of M into disjoint connected sets $L = \{L_\alpha\}_{\alpha \in A}$ with α in some indexing set A, satisfying the following conditions; for every point $p \in M$ there exists an open neighbourhood U of p with a holomorphic coordinate map

$$x = (x_1, ..., x_n) : U \longrightarrow \mathbb{C}^n$$

such that for any $\alpha \in A$, either $L_\alpha \cap U = \emptyset$ or

$$L_\alpha \cap U = \{q \in U | x_i(q) = t_i^\alpha,\ k+1 \leq i \leq n\}$$

where $(t_{k+1}^\alpha, ..., t_n^\alpha) \in \mathbb{C}^n$ depends on α and U.

Each L_α is called a leaf of the foliation. A rank k foliation in a complex manifold of dimension n is sometimes referred to as a codimension $n-k$ foliation. The local behaviour of a rank k foliation can be visualized as the fibres of a projection

$$pr : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-k}$$

where $\mathbb{C}^n$ is considered as $\mathbb{C}^k \oplus \mathbb{C}^{n-k}$ and pr is the projection on the second component. Then for any $c \in \mathbb{C}^{n-k}$, $pr^{-1}(c)$ is a leaf of a rank k foliation $\{L_\alpha^k\}$. Any foliation of rank k is locally isomorphic to $\{L_\alpha^k\}$. The isomorphism is established through the local coordinate system (U, x) which is described above. Such coordinate systems are called *distinguished*. Let (U, x) and (U, y), $y = (y_1, ..., y_n)$ be two distinguished coordinates on U for the foliation $\{L_\alpha\}$. Let g_{xy} be the transition function between x and y

$$x = (x_1, ..., x_n) = g_{xy} \cdot y = (g_{xy}^1 \cdot y, ..., g_{xy}^n \cdot y).$$

Then

$$(\partial g_{xy}^i / \partial y_j) = 0 \quad \text{for} \quad k+1 \leq i \leq n, \quad i \leq j \leq k.$$

i.e. $g_{xy}^i(y_1, ..., y_n) = g_{xy}^i(y_{k+1}, ..., y_n)$ for $i = k+1, ..., n$. This property of foliations lies at the heart of Bott's vanishing theorem which we will mention next and its generalization which we will give in the next chapter.

There is also a vector bundle approach to foliations. Let T be the tangent bundle of M, and let E be a subbundle of T with rank k. In the classical terminology $\mathbb{C}^\infty$ subbundles of T are called smooth distributions. E is called integrable if at each point $p \in M$, there exists a submanifold whose tangent space at p is E_p. Each such submanifold is part of a leaf of a foliation on M. It is easy to see that in this case E is closed under the usual bracket operation. A subbundle of the tangent bundle is called involutive if it is closed under the bracket operation. Integrable and involutive bundles are related to each other by the following classical theorem of Frobenius.

Frobenius Theorem: *A subbundle of the tangent bundle is integrable iff it is involutive.*

One of the questions asked about foliations was, whether given a rank k subbundle E of the tangent bundle T, there is an integrable subbundle F of T such that E can be deformed to F. A necessary condition is given by

Bott's Vanishing Theorem [B2] : *If a subbundle* E *of the tangent bundle* T *is integrable, then the real Pontryagin classes of* T/E *generate a graded ring* $Pont^*(\mathrm{T/E})$ *such that*

$$Pont^i(\mathrm{T/E}) = 0 \quad \text{if} \quad i > 2 \cdot \mathrm{rank}_{\mathbb{R}}(\mathrm{T/E}).$$

Complex Case: *If a holomorphic subbundle* E *of the holomorphic tangent bundle* T *is integrable then the chern classes of* T/E *generate a graded ring* $Chern^*(\mathrm{T/E})$ *such that*

$$Chern^i(\mathrm{T/E}) = 0 \quad \text{if} \quad i > \mathrm{rank}_{\mathbb{C}}(\mathrm{T/E}).$$

The proof of this theorem can be seen as an elegant exploitation of distinguished coordinates, see [B2]. In the next chapter we will give a corollary to this theorem.

In general M may not admit any foliation but one may find a closed subset S of M such that $M - S$ admits a foliation. R. Thom discusses possible ways of choosing such S in [Th]. Obviously if the choice of S is not forced by the foliation on $M - S$ then this does not lead to an interesting mathematical concept. If S in some intrinsic way depends on the foliation on $M - S$ then it is natural to expect that S will reflect some properties of the foliation. To this end we adopt the definition of Baum and Bott for a singular holomorphic foliation;

Definition [BB2]: *A singular holomorphic foliation is a coherent integrable analytic subsheaf of the tangent bundle.*

If F is a singular holomorphic foliation then S is the singular set of F as a coherent sheaf. In the next section we will describe a residue on S coming from the foliation as given by Baum and Bott, [BB2].

2 Baum-Bott Residues

This section presents a summary of Baum and Bott's work on singular foliation residues, [BB2]. At the end of the section a residue for vector bundles is defined which is denoted by BRes and called the generalized Baum-Bott residue.

For any holomorphic vector bundle E we will use the notation that

$$\mathrm{E}' = \text{sheaf of holomorphic sections of E.}$$

Then E'_x will denote the germs of holomorphic sections of E at x.

Let T be the holomorphic tangent bundle of a complex manifold M, and let T be the tangent sheaf. With the above notation $T =$T'. Let ξ be a singular holomorphic foliation of rank k. Then there exists a closed subvariety S of M such that $\xi|M - S$ is locally free, hence there is a vector bundle $\mathrm{E} \to M - S$ of rank k such that

$$\mathrm{E}' = \xi|M - S.$$

To avoid any artificial singularities for this foliation it is assumed that ξ is full, i.e. for every open U in M and for every $\tau \in \Gamma(U,\mathrm{T})$, if $\tau(x) \in \mathrm{E}_x$ for every $x \in U - U \cap S$, then the germ of the holomorphic vector field τ at x is in ξ_x for every $x \in U \cap S$, [BB2, p282].

Baum and Bott compute the chern polynomials $\Phi(\mathrm{T}/\xi)$ in terms of local information at S, where $\Phi \in \mathbb{C}[X_1, \ldots, X_n]$ is a symmetric homogeneous polynomial of degree $d > n - k$. Let Z be a connected component of S,

$$i : Z \longrightarrow M$$

be the inclusion,

$$i_* : H_*(Z;\mathbb{C}) \longrightarrow H_*(M;\mathbb{C})$$

be the natural map induced by i and let

$$PD' : H_*(M;\mathbb{C}) \longrightarrow H^*(M;\mathbb{C})$$

be the Poincare duality map.

For any symmetric homogeneous polynomial $\Phi \in \mathbb{C}[X_1, \ldots, X_n]$ there exists a unique polynomial $\tilde{\Phi}$ such that

$$\tilde{\Phi}(\sigma_1(X_1, \ldots, X_n), \ldots, \sigma_n(X_1, \ldots, X_n)) = \Phi(X_1, \ldots, X_n)$$

where σ_i are the elementary symmetric polynomials. Let $Q = T/\xi$. Then the chern polynomial is defined by

$$\Phi(Q) = \tilde{\Phi}(c_1(Q), \ldots, c_n(Q))$$

where the $c_i(\cdot)$ is the i-th chern class.

Then we have

Theorem (Baum, Bott): *Assume that Z is compact. Then there exists*

$$Res_\Phi(\xi, Z) \in H_{n-d}(Z;\mathbb{C})$$

such that
i) $Res_\Phi(\xi, Z)$ depends only on Φ and the behaviour of ξ near Z.
ii) If M is compact then $\Sigma\, PD' \cdot i_*(Res_\Phi(\xi, Z)) = \Phi(Q)$ where the summation is over all the connected components of Z of S, [BB2, pp312-313].

Let E be a vector bundle of rank k on a complex manifold m of dimension n. Let U be an open subset of M. Let D be a connection of E$|U$ and K a curvature matrix for D. Define $\sigma_1(K), \ldots, \sigma_n(K)$ by the equation

$$\det(I + tK) = 1 + t\sigma_1(K) + \cdots + t^n \sigma_n(K).$$

Each $\sigma_i(K)$ is a $2i$-form on U. It is well known that each $\sigma_i(K)$ is a closed form and defines a unique cohomology class $[\sigma_i(K)]$ in $H^{2i}(M;\mathbb{C})$. By the Chern-Weil Theory

$$[\sigma_i(K)] = \left(\frac{2\pi}{\sqrt{-1}}\right)^i c_i(\mathrm{E}), \quad i = 1, ..., n.$$

For a symmetric homogeneous polynomial $\Phi \in \mathbb{C}[X_1, ..., X_n]$ define

$$\Phi(K) = \tilde{\Phi}(\iota\sigma_1(K), ..., \iota\sigma_n(K))$$

where $\iota = \left(\dfrac{\sqrt{-1}}{2\pi}\right)$. Then

$$[\Phi(K)] = \Phi(\mathrm{E}).$$

Assume that M is compact and there exists a closed subset S of M such that $\mathrm{E}|M-S$ has a connection D_1 with the property that for any symmetric homogeneous polynomial $\Phi(X) = \mathbb{C}[X_1, ..., X_n]$ with $\deg\Phi > d_0$ for some $d_0 > 0$ we have

$$\Phi(K_1) = 0 \ \text{ on } \ M - S$$

where K_1 is the curvature matrix of D_1. Let Σ be a closed subset of M such that S is contained in the interior of Σ and let D_2 be a connection for $\mathrm{E}|\Sigma$. Then there exists a connection D for E on M such that D agrees with D_1 on $M - \Sigma$. To construct such a connection let f be a real valued C^∞ function on M such that f vanishes on a neighbourhood of S and $f = 1$ on $M - S$. Then D is defined as

$$D = fD_1 + (1-f)D_2,$$

which extends D_1, [BB2, p300, Lemma 4.41].

Let K be a curvature matrix for D. Then

$$\Phi(K) = 0 \ \text{ on } \ M - \Sigma.$$

Hence $\Phi(K)$ is a differential form with compact support and defines an element in the cohomology with compact supports,

$$[\Phi(K)] = \Phi(\mathrm{E}) \in H_c^*(M;\mathbb{C}).$$

Let Z be a connected component of S and U an open neighbourhood of Z which deformation retracts to it. Define a residue

$$BRes_\Phi(\mathrm{E}, Z) = (i_*)^{-1} \cdot PD \cdot (\Phi(\mathrm{E})).$$

We will call $BRes$ the *generalized Baum-Bott residue* and use $BRes$ in section 4 to describe the Nash residue. Also note that if F is a coherent sheaf and $\Phi(K)$ is a differential form such that

$$[\Phi(K)] = \Phi(F)$$

and $\Phi(K)$ has compact support in U as above, then $BRes$ is defined for F. In particular if ξ is an integrable coherent subsheaf of T then such a differential form exists for T/ξ, [BB2, p313, (7.11)]. Then note that

$$Res_\Phi(\xi, Z) = BRes_\Phi(T/\xi).$$

Regarding the calculation of Res_Φ, Baum and Bott give the following conjecture:

Rationality Conjecture, [BB2, page 287] *If* $\Phi \in \mathbb{Q}[X_1, \ldots, X_n]$ *and* $\deg \Phi > n - k + 1$*, then*

$$Res_\Phi(\xi, Z) \in H_*(Z, \mathbb{Q}).$$

In the next section we will summarize Suwa's recent contribution to this conjecture and we will relate his result to $BRes$. In section 4 we will give a calculation for Res_Φ when M is a compact Kaehler manifold.

3 Suwa's Work

Let Ω denote the cotangent sheaf of M and let ψ be a subsheaf of Ω. Define

$$\xi = \{\theta \in T |\ \omega(\theta) = 0 \text{ for all } \omega \in \psi\}.$$

Then ξ is an integrable subsheaf of the tangent sheaf iff ψ is closed under the exterior differentiation. Note

$$0 \to \xi \to T \to Q \to 0 \tag{1}$$

where Q is defined as T/ξ. Let Ω_ψ be defined by the exact sequence

$$0 \to \psi \to \Omega \to \Omega_\psi \to 0. \tag{2}$$

Taking the dual, $Hom_{\mathcal{O}}(\cdot, \mathcal{O})$, of (2) we obtain

$$0 \to Hom_{\mathcal{O}}(\Omega_\psi, \mathcal{O}) \to Hom_{\mathcal{O}}(\Omega, \mathcal{O}) \to Hom_{\mathcal{O}}(\psi, \mathcal{O}) \to \eta \to 0 \tag{3}$$

where $\eta = Ext^1_{\mathcal{O}}(\Omega_\psi, \mathcal{O})$. In this sequence

$$\begin{aligned} Hom_{\mathcal{O}}(\Omega_\psi, \mathcal{O}) &= \psi^a = \xi \\ Hom_{\mathcal{O}}(\Omega, \mathcal{O}) &= T \\ Hom_{\mathcal{O}}(\psi, \mathcal{O}) &= \psi^* = \text{dualof } \psi. \end{aligned}$$

Using these identifications (3) can be rewritten as

$$0 \to \xi \to T \to \psi^* \to \eta \to 0. \tag{3'}$$

The kernel of the map $T \to \psi^*$ is ξ by the exactness of (3'). From (1) $T/\xi = Q$, hence these give an exact sequence

$$0 \to Q \to \psi^* \to \eta \to 0. \tag{4}$$

Assume that ψ is locally free of rank $n-k$, where $k = \text{rank}\xi$ and $n = \text{rank}T$, then ψ is called a *foliation of complete intersection type.* In this case the sequence (4) can be interpreted as resolving the coherent sheaf Q into a difference of a vector bundle ψ^* and a skyscraper sheaf η which has support in S, the singular set of ξ.

Let $\sigma(i; X_1, \ldots, X_n)$ be the i-th elementary symmetric polynomial on $X_1, \ldots, X_n$. For each coherent sheaf E let $\sigma(i; E)$ denote the i-th elementary symmetric polynomial on the chern characters of E.

Theorem (Suwa): *Let ψ be a foliation of complete intersection type on M with rank$(\psi) = n - k$. Let Z be a connected component of S, the singular set, and U an open neighbourhood of Z which deformation retracts to Z. Let $\Phi \in [X_1, ..., X_n]$ be a homogeneous symmetric polynomial with degree m such that*

$$\Phi(X_1, ..., X_n) = \sigma(j_1; X_1, ..., X_n) \cdots \sigma(j_r; X_1, ..., X_n)$$

with

$$j_1 + \cdots + j_r = m$$

and

$$j_\nu > n - k \text{ for some } \nu. \qquad (*)$$

Then

$$Res_\Phi(\xi, Z) = (i_*)^{-1} \cdot PD(c_{j_1}(\psi^* - \eta) \cdots c_{j_r}(\psi^* - \eta))$$

where i_* and PD are as defined in section 2.

Proof: For details of the proof we refer to [Su]. Here we will concentrate on the main argument. Let us restrict all the above sequences to U. From the sequence (4) we have, as virtual bundles

$$Q = \psi^* - \eta.$$

Define

$$1 + d_1 + \cdots + d_n = (1 + \sigma(1; \eta) + \cdots + \sigma(n; \eta))^{-1}.$$

Then

$$(1 + \sigma(1; Q) + \cdots + \sigma(n; Q)) = (1 + \sigma(1; \psi^*) + \cdots + \sigma(n; \psi^*))(1 + d_1 + \cdots + d_n).$$

From this it follows that for $j = 1, \ldots, n$ we have

$$\sigma(j; Q) = \sigma(j; \psi^*) + \sigma(j-1; \psi^*)d_1 + \cdots + \sigma(1; \psi^*)d_{j-1} + d_j \qquad (5)$$

and hence

$$\begin{aligned} \Phi(Q) &= \sigma(j_1; Q) \cdots \sigma(j_r; Q) \\ &= \sigma(j_1; \psi^*) \cdots \sigma(j_r; \psi^*) + P \qquad (6) \end{aligned}$$

where P is a polynomial in

$$\sigma(1;\psi^*),\ldots,\sigma(n-k;\psi^*),d_1,...,d_n$$

such that each monomial of P has at least one d_i as a nontrivial factor for $1 \leq i \leq n$. Since $j_\nu > n-k$ for some ν by condition $(*)$, we have $\sigma(j_\nu;\psi^*) = 0$ and hence

$$\Phi(Q) = P.$$

By applying $(i_*)^{-1} \cdot PD$ to both sides of this equation we get the required result.
QED

In this proof since each d_i already represents classes in $H^*(Z;\mathbb{C})$, then P is also in $H^*(Z;\mathbb{C})$. Hence the rationality conjecture directly follows in this case because one did not have to calculate residues; $Res_\Phi(\xi, Z)$ is given by the image of $\Phi(\psi^* - \eta)$ in $H^*(U;Q)$ see [Su, corollary 3.8].

REMARK: The set up being as in Suwa's theorem drop the condition $(*)$, then

$$Res_\Phi(\xi;Z) = BRes_\Phi(\psi^*,Z) + (i_*)^{-1} \cdot PD(P)$$

where P is as in equation (6).
This can be seen as follows; since Φ is defined as

$$\Phi(X_1,...,X_n) = \sigma(j_1;X_1,...,X_n)\cdots\sigma(j_r;X_1,...,X_n)$$

the first term on the left hand side of equation (6) is $\Phi(\psi^*)$. Applying $(i_*)^{-1} \cdot PD$ to both sides of equation (6) gives

$$(i_*)^{-1} \cdot PD(\Phi(Q)) = (i_*)^{-1} \cdot PD(\Phi(\psi^*)) + (i_*)^{-1} \cdot PD(P)$$

where by definition

$$\begin{array}{rcl} (i_*)^{-1} \cdot (\Phi(Q)) & = & Res_\Phi(\xi, Z) \\ \text{and } (i_*)^{-1} \cdot (\Phi(\psi^*)) & = & BRes_\Phi(\psi^*, Z). \end{array}$$

4 Nash Residue and Reduction

This section culls the results of the previous sections to give a new approach to the calculation of Baum-Bott residues for singular holomorphic foliations on compact Kaehler manifolds, where the Nash blow up gives a smooth manifold. It will be shown that if in addition the foliation is defined by a bundle morphism then the Rationality Conjecture of Baum and Bott holds.

First the Nash Residue is defined for general singular holomorphic foliations. Let M be a compact Kaehler manifold of dimension n with tangent sheaf T. Let $\mathcal{F}$ be an integrable full coherent subsheaf of T with rank$\mathcal{F} = k$, such that the Nash Blow-up N of M with respect to $\mathcal{F}$ and T is smooth. On N there is a short exact sequence of vector bundles

$$0 \longrightarrow \tau \longrightarrow \mathbb{C}^n \longrightarrow W \longrightarrow 0$$

where τ, the tautological bundle, restricts to the tautological bundle of each fibre $\pi^{-1}(x) \cong G(k,n)$, $x \in M$. $\mathbb{C}^n$ is the trivial n-bundle and W is the quotient bundle which restricts to the universal quotient bundle of each fibre. This exact sequence restricts to the following short exact sequence on N

$$0 \longrightarrow \tau \longrightarrow \pi^*T \longrightarrow W \longrightarrow 0$$

where T is the tangent bundle on M and π^*T denotes the pull back bundle on N. We use the same notation τ and W for the restriction of these bundles to N since we will be working on N from now on and there will be no ambiguity about the base space.

Let S be the singular set of $\mathcal{F}$. On $M - S$ there is a unique holomorphic vector bundle Y such that $Y' = Q|M - S$. By Bott's vanishing theorem Y has a connection D_1 such that if K_1 is the corresponding curvature matrix and $\Phi \in \mathbb{C}[X_1, ..., X_n]$ is a symmetric homogeneous polynomial with $\deg \Phi > n - k$ then

$$\Phi(K_1) = 0 \text{ on } M - S.$$

Since $\pi^*(Y)$ and W agree on $N - \pi^{-1}(S)$, the connection D_1 of Y pulls back to a connection $\pi^* D_1$ of $W|N - \pi^{-1}(S)$. There exist a connection D of W on N and a compact subset Σ of N which contains $\pi^{-1}(S)$ in its interior such that

$$D|N - \Sigma = \pi^* D_1 | N - \Sigma,$$

[BB2,p300]. Let K_W be the corresponding curvature matrix. Then as above

$$\Phi(K_W) = 0 \text{ on } N - \Sigma,$$

i.e. $\Phi(K_W)$ is a differential form on N with compact support.

Let Z be a connected component of S, U be an open neighbourhood of Z that deformation retracts to Z,

$$\delta : U \longrightarrow Z,$$

and choose Σ such that the connected component of Σ that contains $\pi^{-1}(Z)$ in its interior is contained in $\pi^{-1}(U)$. Then $\Phi(K_W)|\pi^{-1}(U))$ is a differential form on $\pi^{-1}(U)$ with compact support. Hence the chern polynomial $\Phi(W|\pi^{-1}(U))$ is a cohomology class of U with compact support.

Definition: *The Nash Residue of $\mathcal{F}$ on Z is defined as*

$$NRes_\Phi(\mathcal{F}, Z) = \delta_* \cdot \pi * \cdot PD_{\pi^{-1}(U)}\Phi(W|\pi^{-1}(U)).$$

Remark: Let V be an open neighbourhood of M' such that V deformation retracts to $\pi^{-1}(Z)$,

$$\rho : V \longrightarrow \pi^{-1}(Z).$$

Then choose U to be $\pi(V)$ and choose δ such that

$$\begin{aligned} \delta(x) &= \pi\rho\pi^{-1}(x) \text{ if } x \in U - S \\ \delta(x) &= \text{ if } x \in S. \end{aligned}$$

Then δ is a deformation retraction and

$$\delta \cdot \pi(y) = \pi \cdot \rho(y) \text{ for } y \in \pi^{-1}(U) = V.$$

At the homology level this implies that

$$\delta_* \cdot \pi_* = \pi_* \cdot \rho_*.$$

Then we have

$$\begin{aligned} NRes_\Phi(\mathcal{F}, Z) &= \delta_* \cdot \pi_* \cdot PD_{\pi^{-1}(U)}\Phi(W|\pi^{-1}(U)) \\ &= \pi_* \cdot \rho_* PD_{\pi^{-1}(U)}\Phi(W|\pi^{-1}(U)) \end{aligned}$$

where

$$\rho_* \cdot PD_{\pi^{-1}(U)}\Phi(W|\pi^{-1}(U)) \in H_*(\pi^{-1}(Z); \mathbb{C})$$

is the $BRes$ of W (see Section 2). Therefore

$$NRes_\Phi(\mathcal{F}, Z) = \pi_* \cdot BRes_\Phi(W, \pi^{-1}(Z)).$$

Let us summarize this set up:

Let M be a compact Kaehler manifold of dimension n. Let E be a vector bundle of rank k and

$$\Psi : \mathrm{E} \longrightarrow T$$

be a bundle morphism of maximal rank.

Suppose that $\Psi(\mathrm{E})$ generates an integrable coherent subsheaf $\mathcal{F}$ of the tangent sheaf of M. Construct the Nash Blow-up

$$\pi : N \longrightarrow M$$

corresponding to $\mathcal{F}$.

Assume that N is smooth. Let S be the singular set of $\mathcal{F}$ and Z a connected component of S.

Lemma 1: *Let $\Psi' : E \to T$ be the map induced by $\Psi : \mathrm{E} \to \mathrm{T}$ at the sheaf level. Then Ψ' is injective.*

Proof: Let U be an open subset of M such that E and T are trivial on U. Let $e_1, \ldots, e_k$ and $t_1, \ldots, t_n$ be local generators for E and T respectively. Then Ψ can be defined in terms of these basis elements as:

$$\Psi(e_i) = f_{i1}t_1 + \cdots + f_{in}t_n, \;\; i = 1, ..., k$$

where $f_{ij} \in \mathcal{O}_U$ are holomorphic functions on U. If h is a section of $\mathrm{E}|U$ then

$$\begin{aligned} h &= h_1e_1 + \cdots + h_ke_k \\ &= (h_1, ..., h_k) \end{aligned}$$

where h_i are holomorphic functions on U. $\Psi(h)$ can then be defined by

$$\begin{aligned} \Psi(h) &= h_1\Psi(e_1) + \cdots + h_k\Psi(e_k) \\ &= (h_1f_{11} + \cdots + h_kf_{k1}, ..., h_1f_{1n} + \cdots + h_kf_{kn}). \end{aligned}$$

If $\Psi(h) = 0$ then in particular $\Psi_x(h(x)) = 0$, i.e.

$$h_1(x)f_{11}(x) + \cdots + h_h(x)f_{ki}(x) = 0 \;\text{ for }\; x \in U \;\text{ and }\; i = 1, ..., k.$$

Since Ψ is injective as a bundle morphism on $U - U \cap S$ then $h(x) = 0$ for $x \in U - U \cap S$, i.e.

$$h_i(x) = 0 \;\text{ for }\; x \in U - U \cap S \;\text{ and }\; i = 1, ..., k.$$

Then h_i are holomorphic functions on U vanishing on the open set $U - U \cap S$ and consequently are identically zero on U. This proves that Ψ' is injective.

QED

Theorem 1:

$$Res_\Phi(\mathrm{E}, Z) = NRes_\Phi(\mathrm{E}, Z) + \pi_*\tilde{\beta}$$

where $\tilde{\beta} \in H_*(\pi^{-1}(Z); \mathbb{C})$ and is calculated by a Grassmann Graph construction. Moreover $NRes_\Phi(E, Z)$ and $\pi_*\tilde{\beta}$ are rational, hence $Res_\Phi(E, Z)$ is rational.

Proof: On M there is the exact sequence of sheaves

$$0 \longrightarrow E \longrightarrow T \longrightarrow Q \longrightarrow 0 \tag{1}$$

where $E \to T$ is induced by Φ. This sequence pulls back to a sequence of sheaves on N

$$0 \longrightarrow \pi^*E \longrightarrow \pi^*T \longrightarrow \pi^*Q \longrightarrow 0. \tag{2}$$

On N there is also the sequence of vector bundles

$$0 \longrightarrow \tau \longrightarrow \pi^*\mathrm{T} \longrightarrow W \longrightarrow o \tag{3}$$

where τ is the tautological bundle and W is the universal quotient bundle.

Let $X = \pi^{-1}(S)$ and W' =the sheaf of holomorphic sections of W. On $N - X$ the sheaves π^*Q and W' are equal. Hence the sequence of sheaves on N

$$0 \longrightarrow \pi^*E \longrightarrow \pi^*T \longrightarrow W' \longrightarrow 0 \tag{4}$$

is exact on $N - X$. The underlying vector bundles of this sequence give a complex of bundles on N

$$0 \longrightarrow \pi^*\mathrm{E} \longrightarrow \pi^*\mathrm{T} \longrightarrow W \longrightarrow 0 \tag{5}$$

which is exact on $N - X$.

Consider on N the virtual bundle γ

$$\gamma = \pi^*\mathrm{T} - \pi^*\mathrm{E} - W.$$

The chern class of this virtual bundle is

$$\begin{aligned} c(\gamma) &= c(\pi^*\mathrm{T} - \pi^*\mathrm{E} - W) \\ &= c(\pi^*(\mathrm{T} - \mathrm{E}) - W) \\ &= c(\pi^*(\mathrm{T} - \mathrm{E}))/c(W). \end{aligned} \tag{6}$$

From the exactness of equation (2) the chern class of π^*Q is

$$c(\pi^*Q) = c(\pi^*(\mathrm{T} - \mathrm{E})) \tag{7}$$

Combining the results of (6) and (7) one gets

$$c(\gamma) = c(\pi^*Q)/c(W). \tag{8}$$

This equation will be used with the Graph construction on (5); construct the Grassmann Graph corresponding to the complex of vector bundles (5) on N,

$$p : G \longrightarrow N. \tag{9}$$

Let $\xi \to G$ be the virtual tautological bundle on G and let $\{Z_\lambda\}_{\lambda\in\mathbb{P}^1}$ be the family of rationally equivalent cycles in G obtained by the Graph construction. Then

$$\xi|Z_0 = p^*(\gamma) \tag{10}$$

since Z_0 is isomorphic to N. The cycle at infinity Z_∞ decomposes as

$$Z_\infty = N_* + Z_* \tag{11}$$

where N_* is bimeromorphic to N and Z_* is the fibre above S. It is known that

$$\xi | N_* = 0 \tag{12}$$

[BFM,p122], [Fu,pp340-341].

Since Z_0 is rationally equivalent to Z_∞ we get the following equalities

$$\begin{aligned} c_i(\xi) \cap Z_0 &= c_i(\xi) \cap Z_\infty \\ &= c_i(\xi) \cap N_* + c_i(\xi) \cap Z_* \\ &= c_i(\xi) \cap Z_*, \quad i > 0 \end{aligned} \tag{13}$$

where the last equation follows from (12) and $c_i(\xi) \cap Z_* \in H_*(Z_*; \mathbb{C})$. Using (10) and (13) gives

$$\begin{aligned} c_i(\gamma) \cap [N] &= p_*(c_i(\xi) \cap Z_0) \\ &= p_*(c_i(\xi) \cap Z_*) \end{aligned} \tag{14}$$

where $p_*(c_i(\xi) \cap Z_*) \in H_*(X; \mathbb{C})$. Define a localized chern class $c_X^i \in H_*(X; \mathbb{C})$ as

$$c_X^i = p_*(c_i(\xi) \cap Z_*), \quad i > 0. \tag{15}$$

Equation (14) can be rewritten in this notation

$$c_i(\gamma) \cap [N] = c_X^i, \quad i > 0. \tag{14'}$$

The total chern class of γ is then given by

$$\begin{aligned} c(\gamma) \cap [N] &= (1 + c_1(\gamma) + \cdots + c_n(\gamma)) \cap [N] \\ &= [N] + c_X^1 + \cdots + c_X^n \end{aligned} \tag{16}$$

where the last equation is written using (14') and $[N]$ is the fundamental cycle of N. Substitute equation (8) to the LHS of equation (16)

$$\{c(\pi^* Q)/c(W)\} \cap [N] = [N] + \sum_{i=1}^{n} c_X^i. \tag{16'}$$

Cap both sides of (16') by $c(W)$

$$c(W) \cap (\{c(\pi^* Q)/c(W)\} \cap [N]) = c(W) \cap [N] + \sum_{i=1}^{n} c(W) \cap c_X^i. \tag{17}$$

The LHS of (17) can be written:

$$(c(W) \cup \{c(\pi^* Q)/c(W)\}) \cap [N] \tag{18}$$

[Sp, p254, (18)]. In this expression the $c(W)$'s cancel each other since the total chern class of a vector bundle is invertible. Hence (17) takes the form

$$c(\pi^*Q) \cap [N] = c(W) \cap [N] + \sum_{i=1}^{n} c(W) \cap c_X^i. \tag{19}$$

To simplify the notation define $\beta_i \in H_{2n-2i}(X;\mathbb{C})$ as

$$\beta_1 + \cdots + \beta_n = \sum_{i=1}^{n} c(W) \cap c_X^i. \tag{20}$$

Then from (19) and (20) the dual of the i-th chern class of π^*Q becomes:

$$c_i(\pi^*Q) \cap [N] = c_i(W) \cap [N] + \beta_i, \quad i = 1, ..., n. \tag{21}$$

This will be used to calculate $\Phi(\pi^*Q)$ which is defined as

$$\Phi(\pi^*Q) = \tilde{\Phi}(c_1(\pi^*Q), ..., c_n(\pi^*Q))$$

where $\tilde{\Phi}$ was defined before. To calculate $\Phi(\pi^*Q)$ first assume that

$$\Phi(\pi^*Q) = c_i(\pi^*Q)c_j(\pi^*Q) \tag{22}$$

for some $i, j, \quad 0 < i, j < n$. Cap both sides of (22) by $[N]$

$$\Phi(\pi^*Q) \cap [N] = \{c_i(\pi^*Q)c_j(\pi^*Q)\} \cap [N]. \tag{23}$$

The RHS can be written as

$$\{c_i(\pi^*Q)c_j(\pi^*Q)\} \cap [N] = (c_i(\pi^*Q) \cap [N]) \cdot ((c_j(\pi^*Q) \cap [N]) \tag{24}$$

where $\cdot$ is the cycle intersection, [GH,p59]. The RHS (24) can be rewritten using (21)

$$\begin{aligned} \text{RHS}(24) &= (c_i(W) \cap [N] + \beta_i) \cdot ((c_j(W) \cap [N] + \beta_j) \\ &= (c_i(W) \cap [N]) \cdot ((c_j(W) \cap [N]) + (c_i(W) \cap [N]) \cdot \beta_j \\ &\quad + \beta_i \cdot (c_j(W) \cap [N]) + \beta_i \cdot \beta_j. \end{aligned} \tag{25}$$

To shorten the notation define $\beta_{ij} \in H_*(X;\mathbb{C})$ as

$$\beta_{ij} = (c_i(W) \cap [N]) \cdot \beta_j + \beta_i \cdot (c_j(W) \cap [N]) + \beta_i \cap \beta_j. \tag{26}$$

That β_{ij} is a cycle in X follows from the fact that each β_i is a cycle in X, see the definition of β_i in (20). Then (25) can be rewritten as

$$\text{RHS}(24) = (c_i(W) \cap [N]) \cdot (c_j(W) \cap [N]) + \beta_{ij}. \tag{25'}$$

Use the same computation as in (24) for the first term on the RHS of (25');

$$(c_i(W) \cap [N]) \cdot (c_j(W) \cap [N]) = \{c_i(W)c_j(W)\} \cap [N] \tag{27}$$

Using the assumption of (22) that

$$\Phi(*) = c_i(*)c_j(*)$$

the RHS of (27) can be written

$$\{c_i(W)c_j(W)\} \cap [N] = \Phi(W) \cap [N]. \tag{28}$$

These calculations can be put together as follows;

$$\begin{aligned}
\Phi(\pi^*Q) \cap [N] &= \{c_i(\pi^*Q) \cdot c_j(\pi^*Q)\} \cap [N] && \text{from 23} \\
&= (c_i(\pi^*Q) \cap [N]) \cdot (c_j(\pi^*Q) \cap [N]) && \text{from 24} \\
&= (c_i(W) \cap [N]) \cdot (c_j(W) \cap [N]) + \beta_{ij} && \text{from 25'} \\
&= \{c_i(W)c_j(W)\} \cap [N] + \beta_{ij} && \text{from 27} \\
&= \Phi(W) \cap [N] + \beta_{ij} && \text{from the definition of } \Phi
\end{aligned}$$

Hence by induction on the size of Φ we obtain for general Φ

$$\Phi(\pi^*Q) \cap [N] = \Phi(W) \cap [N] + \beta \tag{29}$$

where $\beta \in H_*(X; \mathbb{C})$. Apply π_* to both sides of (29),

$$\pi_*(\Phi(\pi^*Q) \cap [N]) = \pi_*(\Phi(W) \cap [N]) + \pi_*\beta. \tag{30}$$

In the following three steps the terms of this equation will be examined and will be shown to be related to Baum-Bott and Nash Residues.

1) Notice that

$$\Phi(\pi^*Q) = \pi^*\Phi(Q). \tag{31}$$

Since each term in $\Phi(\pi^*Q)$ is a product of chern classes of π^*Q and as is well known

$$c(\pi^*Q) = \pi^*c(Q) \tag{32}$$

Then the LHS of (30) can be simplified;

$$\begin{aligned}
\pi_*(\Phi(\pi^*Q) \cap [N]) &= \pi_*(\pi^*\Phi(Q) \cap [N]) \\
&= \Phi(Q) \cap \pi_*[N] \\
&= \Phi(Q) \cap (\deg \pi)[M] \\
&= \Phi(Q) \cap [M]
\end{aligned} \tag{33}$$

where the first equation follows from (31), the second equation is a property of cap products, [Sp, p254, (61)]. The third equation holds by definition since

$\deg \pi = 1$. Using the Baum-Bott construction more can be said about $\Phi(Q)$. Let $Z_1, ..., Z_m$ be the connected components of S. For each Z_i choose an open neighbourhood U_i of Z_i such that U_i deformation retracts to Z_i

$$\delta_i : U_i \longrightarrow Z_i \tag{34}$$

and $U_i \cap U_j = \emptyset$, $i, j = 1, ..., m$. In each U_i choose a compact set Σ_i such that Z_i is contained in the interior of Σ_i, $i = 1, ..., m$. Let

$$\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_m. \tag{35}$$

There exists a closed differential form ω on M with support on Σ such that

$$[\omega] = \Phi(Q) \tag{36}$$

where $[\cdot]$ denotes the cohomology element defined by that form, [BB2, p312-313]. Let ω_i be defined as follows:

$$\omega_i | U_i = \omega | U_i \quad \text{and} \quad \omega_i | M - U_i = 0, \quad i = 1, ..., m. \tag{37}$$

Then $\omega = \omega_1 + \cdots + \omega_m$ and $[\omega] = [\omega_1] + \cdots + [\omega_m]$, hence

$$[\omega] \cap [M] = [\omega_1] \cap [M] + \cdot + [\omega_m] \cap [M]. \tag{38}$$

But since ω_i is a differential form whose support is compact and is in U_i then

$$[\omega_i] \cap [M] = [\omega_i] \cap [U_i], \quad i = 1, ..., m. \tag{39}$$

Then the LHS of (30) can be written, using (33), (36) and (38) as

$$\begin{aligned} \pi_*(\Phi(\pi^* Q) \cap [N]) &= \Phi(Q) \cap [N] && \text{from } 33 \\ &= [\omega] \cap [M] && \text{from } 36 \\ &= [\omega_1] \cap [M] + \cdots + [\omega_m] \cap [M]. && (40) \end{aligned}$$

Substituting (39) into (40) gives

$$\pi_*(\Phi(\pi^* Q) \cap [N]) = [\omega_1] \cap [U_1] + \cdots + [\omega_m] \cap [U_m]. \tag{41}$$

Hence the global expression on the left splits up as the sum of local expressions.

2) To calculate the first term on the RHS of (30) first recall that $\mathrm{T}/\Psi(\mathrm{E})$ is a vector bundle on $M - S$ and since $\Psi(\mathrm{E})|M - S$ is integrable $\mathrm{T}/\Psi(\mathrm{E})$ has a basic connection, i.e. if $K = K(D)$ is the curvature matrix for this connection on $M - S$, then the differential form $\Phi(K) = 0$ on $M - S$ for $\deg \Phi > n - k$, [BB2, p295].

Since $W|N - X = \pi^*(\mathrm{T}/\Psi(\mathrm{E}))|N - X$, pull the connection D by π^* to a connection on $W|N - X$. Then there exists a connection $\tilde{D}$ for W on N such that $\tilde{D}$ agrees with $\pi^* D$ on $N - \pi^{-1}(\Sigma)$, [BB2, p330, (4.41)]. If $\tilde{K} = \tilde{K}(\tilde{D})$ is the curvature matrix for $\tilde{D}$, then

$$\Phi(\tilde{K}) = 0 \quad \text{on} \quad N - \pi^{-1}(\Sigma) \tag{42}$$

since D and $\tilde{D}$ agree on $N - \pi^{-1}(\Sigma)$. From (42) it follows that we can define closed forms Φ_i on N by

$$\Phi_i | \pi^{-1}(U_i) = \iota \Phi(\tilde{K}) | \pi^{-1}(U_i), \quad \text{where} \quad \iota = (1/2\pi\sqrt{-1})^{\deg \Phi} \tag{43a}$$

and

$$\Phi_i | N - \pi^{-1}(U_i) = 0, \quad i = 1, ..., m. \tag{43b}$$

Then

$$\iota \Phi(\tilde{K}) = \Phi_1 + \cdots + \Phi_m \tag{44}$$

where ι is as in (43a). By definition

$$\iota[\Phi(\tilde{K})] = \Phi(W) \tag{45}$$

Using (44) and (45) together gives

$$\begin{aligned} \Phi(W) &= \iota[\Phi(\tilde{K})] \\ &= [\Phi_1] + \cdots + [\Phi_m]. \end{aligned} \tag{46}$$

Cap both sides of (46) by $[N]$,

$$\Phi(W) \cap [N] = [\Phi_1] \cap [N] + \cdots + [\Phi_m] \cap [N]. \tag{47}$$

Each Φ_i is a closed form with compact support, whose support lies in $\pi^{-1}(U_i) = V_i$, $i = 1, ..., m$. Hence

$$[\Phi_i] \cap [N] = [\Phi_i] \cap [V_i], \quad i = 1, ..., m. \tag{48}$$

Then (47) can be written as

$$\Phi(W) \cap [N] = [\Phi_1] \cap [V_1] + \cdots + [\Phi_m] \cap [V_m]. \tag{49}$$

Notice that $\tilde{D}|V_i$ is a connection for $W|V_i$ and $\tilde{K}|V_i$ is a curvature matrix for $W|V_i$. If $\deg \Phi > n - k$, then

$$\Phi(\tilde{K}|V_i) \quad \text{has compact support in} \quad V_i$$

and

$$\iota[\Phi(\tilde{K}|V_i)] = \Phi(W|V_i). \tag{50}$$

Using (50), (43a) and (43b) gives

$$\Phi(W|V_i) = [\Phi_i]. \tag{51}$$

Each $[\Phi_i]$ is a cohomology class with support in $\pi^{-1}(\Sigma_i)$, hence has compact support and is the chern class of a vector bundle, $W|V_i$, $i = 1, ..., n$. Then $[\Phi_i]$ is in the image of cohomology with rational coefficients on V_i. Using (49) and (51) gives

$$\Phi(W) \cap [N] = \Phi(W|V_1) \cap [V_1] + \cdots + \Phi(W|V_m) \cap [V_m]. \tag{52}$$

Apply π_* to both sides of (52),

$$\pi_*(\Phi(W) \cap [N]) = \Sigma_{i=1}^m \pi_*(\Phi(W|V_i) \cap [V_i]) \tag{53}$$

where $\pi_*(\Phi(W|V_i) \cap [V_i]) \in H_*(U_i; \mathbb{C})$ for $i = 1, ..., m$.

3) The last element on the right hand side of (30) is $\pi_*\beta$. From the construction of β it follows that β naturally splits as

$$\beta = \tilde{\beta_1} + \cdots + \tilde{\beta_m} \tag{54}$$

where $\tilde{\beta_i} \in H_*(X; \mathbb{C})$ for $i = 1, ..., m$. Since $\tilde{\beta_i}$ is obtained by capping chern classes of W with the residue obtained from Grassmann Graph and by intersection of these, it follows that $\tilde{\beta_i}$ is in the image of homology with rational coefficients in X, see equations (20), (26) and (29).

Apply π_* to both sides of (54)

$$\pi_*\beta = \pi_*\tilde{\beta_1} + \cdots + \pi_*\tilde{\beta_m} \tag{55}$$

where $\pi_*\tilde{\beta_i} \in H_*(Z_i; \mathbb{C})$, $i = 1, ..., m$.

This completes the examination of the terms of (30).

Putting equations (41), (53) and (55) into (30) gives

$$\Sigma_{i=1}^m [\omega_i] \cap [U_i] = \Sigma_{i=1}^m \pi_*(\Phi(W|V_i) \cap [V_i]) + \Sigma_{i=1}^m \pi_*\tilde{\beta_i}. \tag{56}$$

From (56) we can now write

$$[\omega_i] \cap [U_i] = \pi_*(\Phi(W|V_i) \cap [V_i]) + \pi_*\tilde{\beta_i}, \tag{57}$$

since each summand in (56) is in $H_*(U_i; \mathbb{C})$, $i = 1, ..., m$.

The deformation retraction δ_i of (34) induces an isomorphism

$$\delta_{i*} : H_*(U_i; \mathbb{C}) \longrightarrow H_*(Z_i; \mathbb{C}) \tag{58}$$

for $i = 1, ..., m$. Apply δ_{i*} to both sides of (57)

$$\delta_{i*}([\omega_i] \cap [U_i]) = \delta_{i*}\pi_*(\Phi(W|V_i) \cap [V_i]) + \delta_{i*}\pi_*\tilde{\beta}_i \tag{59}$$

for $i = 1, ..., m$. For the first term on the LHS of (59) we have

$$\delta_{i*}([\omega_i] \cap [U_i]) = Res_\Phi(\mathrm{E}, Z_i), \quad i = 1, ..., m \tag{60}$$

[BB2, p313, (7.14)]. The first term on the RHS of (59) is the Nash Residue by definition

$$\delta_{i*}\pi_*(\Phi(W|V_i) \cap [V_i]) = NRes_\Phi(\mathrm{E}, Z_i), \quad i = 1, ..., m. \tag{61}$$

Since $\pi_*\tilde{\beta}_i$ is already in $H_*(Z_i; \mathbb{C})$ by (55), δ_{i*} does not change it;

$$\delta_{i*}\pi_*\tilde{\beta}_i = \pi_*\tilde{\beta}_i, \quad i = 1, ..., m. \tag{62}$$

Substituting (60), (61) and (62) into (59) gives

$$Res_\Phi(\mathrm{E}, Z_i) = NRes_\Phi(\mathrm{E}, Z_i) + \pi_*\tilde{\beta}_i \tag{63}$$

for $i = 1, ..., m$ as required.

By appealing to the discussions that follow equations (51) and (54) we conclude that the RHS of (63) is rational, and hence the LHS is rational,

$$Res_\Phi(\mathrm{E}, Z_i) \in H_*(Z_i; \mathbb{Q}), \quad i = 1, ..., m.$$

QED

REMARKS: 1) Let M be a compact complex manifold with a positive line bundle. Then M is algebraic by Kodaira embedding theorem, [GH, p181]. Hence N, being a subvariety of $M \times G(k, T)$, is also algebraic. On algebraic manifolds coherent sheaves have global syzygies, [GH, p701]. Then $\pi_*(F)$ will have global syzygies on N assuming that the Nash blow-up N is smooth. Thus theorem 1 will hold for rich foliations on algebraic manifolds for which the Nash Blow-up is smooth without the further assumption that F be generated by a bundle morphism.

2) If F is generated by a bundle morphism $\Psi : \mathrm{E} \to \mathrm{T}$ but rank$\mathrm{E} >$ rank$F$, then Ψ' is not injective. To make theorem 1 work in this case some restrictions must be imposed on the kernel of Ψ. For example for every connected component Z of S assume that there exist an open neighbourbood U of Z and a vector bundle H on U with a bundle morphism

$$\eta : H|U - Z \longrightarrow \mathrm{E}|U - Z$$

such that η is injective. Then theorem 1 holds for F. Note that η need not be defined on all of U but H should be defined on U to ensure the construction of Grassmann Graph.

3) Notice that theorem 1 holds for a subclass of rich foliations. Call a rich foliation *very rich* if for every connected component Z of S there exists an open neighbourbood U of Z such that to $F|U$ there exists a complex of $\mathcal{O}$-modules on U which give a locally free resolution on $U - Z$. Then theorem 1 holds for very rich foliations for which the Nash Blow-up is smooth. It is natural to conjecture that all rich foliations are very rich.

CHAPTER 4
OBSTRUCTION CLASSES

0 Introduction

This final chapter pursues a problem that arises when singular holomorphic foliations are considered as integrable images of bundle morphisms. Section 1 defines obstruction classes in terms of Pontryagin classes which obstruct the imbedding of a vector bundle into the tangent bundle. Other topological obstructions in terms of Stiefel-Whitney classes can be found in the literature, in particular see Sundararaman [Sr]. Section 2 very briefly summarizes immediate future research projects to which this work leads. We propose to study the problem of Riemann-Roch as explained in section 2 as a consequence of this work.

1 Obstruction Classes

Most well known examples of singular holomorphic foliations are meromorphic vector fields. A meromorphic vector field is defined as follows: let M be a complex manifold of dimension n with tangent bundle T and let Ł be a line bundle on M. Assume that there is a bundle morphism Ψ

$$\Psi : \text{Ł} \longrightarrow \mathrm{T}.$$

Ψ is called a meromorphic vector field. $\Psi(\text{Ł})$ generates a 1 dimensional subsheaf of the tangent sheaf T. By dimension considerations this coherent subsheaf is integrable, therefore it defines a singular foliation, the singularity set being S, where

$$S = \{x \in M | \Psi_x = 0\}.$$

To generalize this concept let E be a vector bundle of rank k on M and assume that there is a morphism

$$\beta : \mathrm{E} \longrightarrow \mathrm{T}.$$

If $\beta(\mathrm{E})$ is integrable then it defines a singular foliation whose singularity set is

$$S = \{x \in M | \mathrm{rank}(\beta_x) < \max.\mathrm{rank}(\beta)\}$$

If $S = \emptyset$ then $\beta(\mathrm{E})$ defines a foliation if it is integrable. If $\max.\mathrm{rank}(\beta) = \mathrm{rank}(\mathrm{E})$ and $S = \emptyset$ then β is an imbedding of E into T. For an arbitrary E clearly no such β exists. This section answers a natural question: "Are there differential geometric obstruction classes for the existence of an imbedding $\beta : \mathrm{E} \to \mathrm{T}$?" For an answer see theorem 2.

First in the following theorem we collect a few facts about splitting manifolds, [H, I.4.2, III.13.2.1].

Theorem (Hirzebruch) : *Let* E *be a vector bundle of rank k on a complex manifold M with tangent bundle* T. *There exists a complex manifold M_s and a holomorphic map*

$$\Phi : M_s \longrightarrow M$$

with the following properties:
i) M_s is a fibre bundle over M with the flag manifold $F(k) = GL(k, \mathbb{C})/\Delta(k, \mathbb{C})$ as fibre, where $\Delta(k, \mathbb{C})$ is the subgroup of $GL(k, \mathbb{C})$ consisting of triangular matrices.
ii) $\Phi^*\mathrm{E}$ *splits as a sum of line bundles on M_s.*
iii) $\Phi^*\mathrm{T}$ *is a quotient bundle of* T_s, *the tangent bundle of M_s;*

$$\mathrm{T}_s = (\Phi^*\mathrm{T}) \oplus E^{\Delta}$$

where E^{Δ} is the bundle along the fibres. Here the direct sum need not be holomorphic, it is in general a C^{∞} direct sum.
iv) The map induced by Φ

$$\Phi^* : H^*(M; \mathbb{C}) \longrightarrow H^*(M_s; \mathbb{C})$$

is a monomorphism.

Remark: To construct M_s, consider the complex analytic principle bundle

$$F \longrightarrow M$$

associated to E. Then

$$M_s = F/\Delta(k, \mathbb{C}).$$

The proof of theorem 2 will need a generalized version of Bott's vanishing theorem, which is given next; assume that M is a complex manifold of dimension $n + m$ with tangent bundle T.

Theorem 1: *Let* $\mathrm{T} = A \oplus B$ and E *be a subbundle of A. If* E *and B are integrable, then the graded Chern ring $Chern^*(A/\mathrm{E})$ vanishes beyond the corank of* E *in A, i.e.*

$$Chern^i(A/\mathrm{E}) = 0 \quad \text{if} \quad i > \mathrm{rank}A - \mathrm{rank}\mathrm{E}.$$

Proof: Let $\mathrm{rank}A = n$, $\mathrm{rank}B = m$ and $\mathrm{rank}\mathrm{E} = k$. It suffices to show that if P is a symmetric, homogeneous ad-invariant polynomial on $GL(k, \mathbb{C})$ of degree greater than $n - k$, then P applied to a curvature matrix of $(A/\mathrm{E})^*$ is zero.

Let $\{U\}$ be an open covering of M such that all the above bundles are trivial on each U, and there is a partition of unity $\{\lambda_U\}$ corresponding to this covering. On U let

$$x_1^U, \ldots, x_n^U, y_1^U, \ldots, y_m^U$$

be local coordinates such that

$$\begin{array}{ll} A^* & \text{is generated by} \quad dx_1^U, \ldots, dx_n^U \\ B^* & \text{is generated by} \quad dy_1^U, \ldots, dy_m^U. \end{array}$$

Let V be another element of $\{U\}$. Then similarly there are coordinates

$$x_1^V, \ldots, x_n^V, y_1^V, \ldots, y_m^V$$

On V such that

$$\begin{array}{ll} A^* & \text{is generated by} \quad dx_1^V, \ldots, dx_n^V \\ B^* & \text{is generated by} \quad dy_1^V, \ldots, dy_m^V. \end{array}$$

If $U \cap V \neq \emptyset$ then there is a transition function h_{UV}^A for A^* such that

$$(dx_1^U, ..., dx_n^U) = h_{UV}^A(dx_1^V, \ldots, dx_n^V, dy_1^V, \ldots, dy_m^V).$$

B is integrable so by Frobenius

$$h_{UV}^A(dx_1^V, \ldots, dx_n^V, dy_1^V, \ldots, dy_m^V) = h_{UV}^A(dx_1^V, \ldots, dx_n^V).$$

Since E is integrable we may assume that the covering is fine enough so that E* is generated by

$$dx_{n-k+1}^U, \ldots, dx_n^U \quad \text{on} \quad U$$

and by

$$dx_{n-k+1}^V, \ldots, dx_n^V \quad \text{on} \quad V$$

Then $(A/\mathrm{E})^*$ is generated by

$$dx_1^U, \ldots, dx_{n-k}^U \quad \text{on} \quad U$$

and by

$$dx_1^V, \ldots, dx_{n-k}^V \quad \text{on} \quad V.$$

If $U \cap V \neq \emptyset$ then there is a transition function g_{UV} such that

$$(dx_1^U, \ldots, dx_{n-k}^U) = g_{UV}(dx_1^V, \ldots, dx_n^V).$$

E is integrable so by Frobenius

$$g_{UV}(dx_1^V, \ldots, dx_n^V) = g_{UV}(dx_1^V, \ldots, dx_{n-k}^V).$$

Let

$$dx^U = (dx_1^U, \ldots, dx_{n-k}^U) \quad \text{and} \quad dx^V = (dx_1^V, \ldots, dx_{n-k}^V).$$

From here on the proof of Bott's vanishing theorem applies, see [B1]. For completeness we include the main parts of the proof.

Let D_U be a connection for $(A/\mathrm{E})^*$ on U defined as

$$D_U \; dx_i^U = 0 \quad i = 1, \ldots, n-k.$$

Then

$$D = \Sigma_U \lambda_U D_U$$

is a connection for $(A/\mathrm{E})^*$. The associated connection matrix θ_U on U is calculated as follows;

$$\begin{aligned} \theta_U = D\; dx^U &= \Sigma_V \lambda_V D_V dx^U \\ &= \Sigma_V \lambda_V D_V (g_{UV} dx^V) \\ &= \Sigma_V \lambda_V (dg_{UV} dx^V + g_{UV} D_V dx^V) \\ &= \Sigma_V \lambda_V dg_{UV} dx^V \\ &= \Sigma_V \lambda_V dg_{UV} g_{VU} dx^U. \end{aligned}$$

We then have to investigate the nature of dg_{UV} to find where the curvature matrix $d\theta_U - \theta_U \wedge \theta_U$ lies. Differentiating both sides of

$$dx^U = g_{UV} dx^V$$

gives

$$0 = dg_{UV} dx^V$$

which implies that dg_{UV} lies in the ideal generated by

$$dx_1^V, \ldots, dx_{n-k}^V.$$

Consequently the curvature matrix lies in the same ideal. Any i-fold product of this ideal with $i > n-k$ is clearly zero. Hence the theorem. **QED**

Notice that when $B = 0$ this theorem reduces to Bott's vanishing theorem. Also note that the above proof shows that if $Pont^*(A/\mathrm{E})$ is the graded Pontryagin ring of A/E generated by the real Pontryagin classes of A/E then

$$Pont^i(A/\mathrm{E}) = 0 \quad \text{if} \quad i > 2(\mathrm{rank}_{\mathbb{R}} A - \mathrm{rank}_{\mathbb{R}} \mathrm{E}).$$

Before defining obstruction classes let us develop some notation. Let

$$a = (a_1, \ldots, a_n)$$

be an n-tuple of nonnegative integers and define

$$|a| = a_1 + 2a_2 + \cdots + na_n.$$

For any vector bundle E, define

$$c^a(\mathrm{E}) = (c_1(\mathrm{E}))^{a_1} \cdots (c_n(\mathrm{E}))^{a_n}$$

and

$$p^a(\mathrm{E}) = (p_1(\mathrm{E}))^{2a_1} \cdots (p_n(\mathrm{E}))^{2a_n}$$

where $c_i(\cdot)$ is the i-th Chern class in $H^i(M;\mathbb{C})$ and $p_i(\cdot)$ is the i-th Pontryagin class in $H^{2i}(M;\mathbb{C})$. Also recall that for any two vector bundles E and F, the Chern class of the virtual bundle $\mathrm{E} - \mathrm{F}$ is defined as

$$c(\mathrm{E} - \mathrm{F}) = (c(\mathrm{E})/c(\mathrm{F})).$$

Theorem 2: *Let* E *be a vector bundle of rank k on a complex manifold M of dimension n with tangent bundle* T*. If* E *can be imbedded into* T *then the following obstruction classes in the cohomology of the splitting manifold M_s are zero:*

$$p^a(\Phi^*\mathrm{T} - \mathrm{L}_i) = 0, \quad i = 1, \ldots, k, \quad |a| = 2n,$$

where $\Phi : M_s \to M$ is the natural projection, and L_i are line bundles such that

$$\Phi^*\mathrm{E} = \mathrm{L}_1 \oplus \cdots \oplus \mathrm{L}_k \quad \text{on} \quad M_s.$$

Proof: Recall that $T_s = \Phi^*T \oplus E^\Delta$ where E^Δ is the bundle along the fibres on M_s and hence is integrable. If E can be imbedded into T then

$$\Phi^*\mathrm{E} = \mathrm{L}_1 \oplus \cdots \oplus \mathrm{L}_k$$

can be imbedded into $\Phi^*\mathrm{T}$ and hence each L_i can be imbedded into $\Phi^*\mathrm{T}$. By dimension considerations each L_i is integrable in T_s. By theorem 1 the Pontryagin rings $Pont^*(\Phi^*\mathrm{T}/\mathrm{L}_i)$ vanish above twice the corank of L_i in $\Phi^*\mathrm{T}$,

$$Pont^i(\Phi^*\mathrm{T}/\mathrm{L}_i) = 0 \quad \text{if} \quad i > 2(n-1).$$

This then completes the proof. **QED**

In particular observe that if there are classes $\gamma_i \in H^*(M;\mathbb{C})$ such that $\Phi^*(\gamma_i) = p_1(\mathrm{L}_i)$, then the graded ring P^* generated in $H^*(M;\mathbb{C})$ by $\{\gamma_1, \ldots, \gamma_k\}$ and the

Pontryagin classes of T vanish in the top dimension. This is because Φ^* is a monomorphism at the cohomology level.

Example: Let M be a complex manifold of dimension n. Let

$$V = M \times \mathbb{C}, \quad \text{and} \quad \pi : V \longrightarrow M$$

be the projection on the first component. Let L be a line bundle on M. Define two line bundles on V as

$$\begin{array}{rcl} \mathrm{L}_1 & = & \pi^*\mathrm{L} \\ \mathrm{L}_2 & = & \text{the line bundle along the fibres.} \end{array}$$

Then L_2 is the trivial line bundle. Let E be a coherent sheaf on M defined by the presheaf

$$\Gamma(E, U) = \Gamma(\mathrm{L}_1 \oplus \mathrm{L}_2, \pi^{-1}(U))$$

for U open in M. Let T be the tangent sheaf of M. If E can be imbedded into T then

$$c_n(T) - c_{n-1}(T)c_1(T) + \cdots + (-1)^n c_1(E)^n = 0.$$

To see this note that π^*E splits on V and

$$c_1(\pi^*E) = c_1(\mathrm{L}_1 \oplus \mathrm{L}_2) = c_1(\mathrm{L}_1) + c_1(\mathrm{L}_2) = c_1(\mathrm{L}_1),$$

and

$$c^n(\pi^*\mathrm{T} - \mathrm{L}_2) = 0$$

by the meromorphic vector field theorem of Baum and Bott, see [BB2], or see theorem 1 above.

2 Future Research Projects

The results of this work naturally lead to new possibilities which are briefly mentioned here.

i) MacPherson has defined Chern classes for singular varieties using Chern-Mather type of characteristic classes with correction factors, see [Mc]. One interesting problem is to define a local Euler obstruction for a coherent subsheaf F of a locally free sheaf G using the associated Nash Blow-up as MacPherson defines a local Euler obstruction using the Nash construction corresponding to the tangent sheaf of a singular variety. This will define homological Chern classes for F and it will be interesting to check if these classes correspond to the usual Chern classes of F obtained through a resolution by locally free real analytic sheaves.

ii) It was conjectured for some time that the Meromorphic Vector Field theorem of Baum-Bott would imply the Riemann-Roch theorem as the Holomorphic Vector Field theorem of Bott did, see [BB1] and [B2]. It will be interesting to see how far the Grassmann Graph construction can be used towards a settlement of this conjecture. In the algebraic case Baum-Fulton-MacPherson used this construction to prove a Riemann-Roch theorem for singular varieties, see [BFM]. Using the graph of a meromorphic vector field in the compact Kaehler case promises to be the right way to attack the above conjecture.

iii) It will be an interesting problem to concentrate on calculating Baum-Bott residues using the degeneracy cycles of the universal quotient bundle W on the Nash Blow-up N. It is natural to conjecture that the intersection cycles corresponding to $\tilde{\Phi}(c_1(W), \ldots, c_{n-k}(W))$ will be homologous to the sum of some rational cycles that lie in $\pi^{-1}(S)$. This will then solve the Rationality conjecture in the compact Kaehler case.

iv) Using the knowledge that the Nash Blow-up corresponding to coherent subsheaves of a locally free sheaf is analytic one can approach the work of Aznar who in the algebraic category generalizes MacPherson's local Euler obstruction, see [Az]. A future project is to study Aznar's generalization in the analytic case in terms of Segre classes as mentioned by Fulton in his 1983 Regional Conference.

v) The obstruction classes defined in this chapter are open for further investigation. One particular direction to continue is to recover the obstruction classes in terms of the Chern classes of E and T. For this it will be necessary to classify those cases where on M_s the tangent bundle T_s accepts $\Phi^*\mathrm{T}$ as a holomorphic factor in the direct sum $\mathrm{T}_s = \Phi^*\mathrm{T} \oplus \mathrm{E}^{\Delta}$.

References

[AH] M. F. Atiyah & F. Hirzebruch,
Analytic Cycles on Complex Manifolds, Topology 1 (1961) 25-45.

[Az] V. Navarro Aznar,
Sur le Multiplicités de Schubert Locales de Faisceaux Algébriques Cohérents, Compositio Mathematica 48 (1983) 311-326.

[B1] R. Bott,
On a Topological Obstruction to Integrability, Proceedings of Symposia in Pure Mathematics, American Mathematical Society, 16 (1970) 127-131.

[B2] R. Bott,
Vector Fields and Characteristic Numbers, Michigan Mathematical Journal, 14 (1967) 231-244.

[BB1] P. F. Baum & R. Bott,
On the Zeroes of Meromorphic Vector Fields, Essays on Topology and Related Topics, Memoires dédies à Georges de Rham, Springer-Verlag (1970) 29-47.

[BB2] P. F. Baum & R. Bott,
Singularities of Holomorphic Foliations, Journal of Differential Geometry, 7 (1972) 279-342.

[BBc] A. Bialynicki-Birula,
Some Theorems on Actions of Algebraic Groups, Annals of Mathematics, 98 (1973) 480-497.

[BFM] P. F. Baum, W. Fulton & R. MacPherson,
Riemann-Roch for Singular Varieties, Publications Mathématiques IHES, 45 (1975) 101-145.

[BS] A. Borel & J. B. Serre,
Le Théorème de Riemann-Roch, Bulletin de la Société Mathématiques de France, 86 (1958) 97-136.

[C] H. Cartan,
Faisceaux Analytiques Cohèrents, Centro Internazionale Mathematico Estivo, Varenna (1963) 1-88.

[CS1] J. B. Carrell & A. Sommese,
$\mathbb{C}^*$-Actions, Mathematica Scandinavica, 43 (1978) 49-59.

[CS2] J. B. Carrell & A. Sommese,
Some Topological Aspects of $\mathbb{C}^*$-Actions on Compact Kaehler Manifolds, Commentarii Mathematici Helvetici, 54 (1979) 567-582.

[F] G. Fischer,
Complex Analytic Geometry, Lecture Notes in Mathematics, Volume 538, Springer-Verlag, (1976).

[Fu] W. Fulton,
Intersection Theory, Springer-Verlag, (1984).

[G] P. Griffiths,
Two Theorems on Extensions of Holomorphic Mappings, Inventiones Mathematicae, 14 (1971) 27-62.

[GH] P. A. Griffiths & J. Harris,
Principles of Algebraic Geometry, John Wiley and Sons, (1978).

[Gr] J. Giraud,
Mathematical Reviews, 53#13217 (1977) 1846.

[H] F. Hirzebruch,
Topological Methods in Algebraic Geometry, 3rd Edition, Springer-Verlag (1966).

[Ha] R. Hartshorne,
Algebraic Geometry, Graduate Texts in Mathematics 52, Springer-Verlag (1977).

[HR] H. Hironaka & H. Rossi,
On the Equivalence of Imbeddings of Exceptional Complex Spaces, Mathematische Annalen, 156 (1964) 313-333.

[L] H. B. Lawson Jr.,
Foliations, Bulletin of the American Mathematical Society, 80 (1974) 369-418.

[Mc] R. D. MacPherson,
Chern Classes for Singular Algebraic Varieties, Annals of Mathematics, 100 (1974) 423-432.

[N] A. Nobile,
Some Properties of the Nash Blowing-up, Pacific Journal of Mathematics, 60 (1975) 297-305.

[R] R. Remmert,
Projektionen Analytische Mengen, Mathematische Annalen, 130 (1956) 410-441.

[Rc] R. W. Richardson Jr.,
Principal Orbit Types for Reductive Groups Acting on Stein Manifolds, Mathematische Annalen, 208 (1974) 323-331.

[Ri] O. Riemenschneider,
Characterising Moisĕzon Spaces by Almost Positive Coherent Analytic Sheaves, Mathematische Zeitscrift, 123 (1971) 263-284.

[Ro] H. Rossi,
Picard Variety of an Isolated Singular Point, Rice University Studies, 54 (1968) 63-73.

[S] J. P. Serre,
Prolongement de Faisceau Analytiques Cohérent, Annales de L'Institu Fourier, Grenoble, 16 (1966) 363-374.

[So] A. J. Sommese,
Extension Theorems for Reductive Group Actions on Compact Kaehler Manifolds, Mathematische Annalen, 218 (1975) 107-116.

[Sr] D. Sundararaman,
Moduli, Deformations and Classifications of Compact Complex Manifolds, Pitman Publishing Ltd., London, (1980).

[ST] Y. Siu & G. Trautmann,
Gap Sheaves and Extension of Coherent Analytic Subsheaves, Lecture Notes in Mathematics, Volume 172, Springer-Verlag, (1971).

[Su] T. Suwa,
Residues of Complex Analytic Foliation Singularities and the Riemann-Roch Theorem for Embeddings, preprint.

[T] W. Thimm,
Extension of Coherent Analytic Subsheaves, Several Complex Variables 1, Maryland, Lecture Notes in Mathematics, Volume 155, Springer-Verlag, (1970) 191-202.

[Th] R. Thom,
On Singularities of Foliations, Manifolds, Tokyo, (1973) 171-173.