

# Style Goods Pricing with Competition

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## Abstract

We study timing decisions for price change of style goods in the existence of competition. We model as a duopoly two firms which are selling their fixed stocks of an item in a given period of time. Our model differs from existing single-firm inventory models in that we explicitly consider the demand interactions through pricing decisions by two competing firms. Each firm starts with the same initial price, and has the option to decrease the price once during the selling season. The demand for the product at one firm depends on the prices offered by both firms. The price levels and corresponding demand rates are known in advance. The problem for each firm is to decide *when* to decrease its price (mark-down) in order to maximize its revenue. We find the unique equilibrium switching times for both firms. We show that the firm with higher inventory will lower its price first but each firm will sell off its inventory at the end of the horizon. We also analyze the effect of demand interaction on price switching decisions. As the demand for one firm gets more dependent on the price offered by the other firm, the larger firm will tend to lower its price later, while the smaller firm will tend to lower its price earlier. Furthermore, with intense demand interaction, the revenue of the larger firm will increase, while the revenue of the smaller firm will decrease. Although our results hold for general symmetric demand functions, we interpret our results in the context of a linear demand model. We then extend our model to the mark-up problem in which either company has one opportunity to increase its price to a preset level. We apply the mark-up model to a discount fare allocation problem for airlines.

# 1 Introduction and Literature Survey

Selling a fixed amount of inventory over a finite horizon is a problem faced by many companies. Examples include retailers selling fashion or seasonal goods such as ski-wear, airlines selling a fixed number of airplane seats, and hotels selling a fixed number of rooms. One frequently cited reason for the lack of replenishment opportunities during the selling season has been the relatively long replenishment lead times as compared to the length of the selling season. Examples include five months lead time for Liz Claiborne apparel (Dalby and Flaherty 1990) and six to eight months lead time for a major fashion retailer (Gallego and van Ryzin 1994). Pashigian (1988) reports that the mean lead time for an apparel order from a Far Eastern country is 34.7 weeks. Selling season, on the other hand, is at most six months for these fashion goods.

Airlines, hotels and many other service companies have fixed capacities (seats and rooms) and share the same characteristic problems as style goods retailing: the fixed costs in acquiring and disposing additional capacities are prohibitive while the variable costs associated with providing these services and products are very low. This problem is also relevant to companies selling products near the end of their life cycles, for which no further production is planned. In all of these problem contexts, the inventory units unsold at the end of the horizon have little value, or in the case of airline seats and hotel rooms, no value at all.

When the demand is price sensitive, price adjustments are made to maximize a firm's revenue over the horizon. Price adjustments can take the form of pure temporal price discrimination, as is the case for retailers that use mark-ups or mark-downs. Pashigian (1988) reports that the dollar value of total mark-downs as a percentage of dollar sales on all merchandise sold in department stores increased over the years, reaching 16.8% in 1984. Fisher (1994) notes that in 1990 this percentage went up to 25% in 1990.

An indirect way of price adjustment is to first differentiate the product in a costless manner, thereby creating different customer classes. The fixed capacity is then allocated to these customer classes by opening or closing classes as time advances. This latter form of price adjustment is known as yield management and is utilized by many service companies. Smith et. al. (1992) reports that American Airlines increased its revenues by \$ 1.4 billion over a three year period, attributable to yield management. We refer to Weatherford and Bodily (1992) for a review of yield management.

In the economics literature, clearance sales as a means of price adjustment (mark-down) has received some attention. Lazear (1986) develops a model of retailing, in which the decision is the price of the product over a finite number of periods to maximize profit. It is assumed that the consumers are homogeneous

(they have the same valuation of the item) and that the consumers shop at once when the price is declared, which makes the length of each period immaterial. Pashigian (1988) extends Lazear’s model to allow for industry equilibrium and shows that fashion and product variety are the leading reasons for increasing mark-down ratios in recent years in fashion retailing. Pashigian and Bowen (1991) show empirically that uncertainty and price discrimination explain the mark-downs in fashion retailing. van Praag and Bode (1992) incorporate cost of clearance sales to Lazear’s model.

Gallego and van Ryzin (1994) study the pricing decisions of a firm selling a fixed stock of items over a finite horizon with demand being a controlled Poisson process. They show that the optimal profit of the deterministic problem, in which demand rates are assumed to be constant, gives an upper bound for the optimal expected profit. For the continuous price case, fixed-price heuristics are shown to be asymptotically optimal. For the discrete price case, a deterministic solution can be used to develop again asymptotically optimal heuristics. Bitran and Mondschein (1997) present a similar model. Feng and Gallego (1995) derive optimal policy for the two price case. Gallego and van Ryzin (1997) study the multi-product case; they suggest two asymptotically optimal heuristics and apply them in network yield management problems.

While the marketing and economics literature contains much research on the effects of price competition, it has generally assumed that instantaneous and infinitely available supply is always possible, thus ignoring the role of limited inventory supply. On the other hand, the inventory literature has examined single-firm stocking and pricing decisions, but the effect of demand interactions in a multi-firm, competitive market has not been studied.

In this study, we use a demand structure similar to those used in Gallego and van Ryzin (1994, 1997) and Feng and Gallego (1995). Our primary objectives and contribution in this research are to understand how competition impacts the pricing decisions of firms selling fixed inventories, and how increasing degrees of demand interaction affect these decisions. We use a game theoretic formulation to derive the equilibrium policies for two competing firms. Under competition, the demand rate for one product depends on its own price and the price of the other product offered by the competing firm. We assume that these two products are not necessarily identical, and hence even if the prices are different, the company offering a higher price may still face some demand. Varian (1980) makes a distinction between the consumers who have complete information about the prices offered by different firms (the “informed” consumers) and the consumers who know nothing about the price distribution (the “uninformed” consumers) to explain the price dispersion even when the products are identical. We use a linear demand model to determine the demand rates for each firm given both the prices

of both firms. Specifically, we use the linear model that is presented in McGuire and Staelin (1983). We describe the details of this model in Section 2.1.

We assume that both firms start with the same price, and that they have the option to decrease the price (mark-down) or increase the price (mark-up) over the selling horizon. The mark-down problem is typical in retailing, while the mark-up problem frequently occurs in service industries such as airlines and hotels. The price levels and their associated demand rates are publicized and known in advance. The problem for each firm then is to find the timing of its price change so as to maximize its (expected) profits. We model the problem as a non-cooperative game. In Section 2, we describe our mark-down model in detail. First, we describe the demand-price relationship using the linear demand model. We then study the single-firm (monopoly) mark-down problem and analyze the problem in the case of two competing firms (duopoly). Qualitative properties of the equilibrium are also derived. The results are compared with the case of no demand interaction between the firms. Section 3 analyzes the mark-up model and describes the transformation of a fare allocation problem to this mark-up model. We present the conclusions and implications in Section 4.

## 2 The Mark-down Problem

In this section, we study the case when firms start with a high initial price and switch to a lower price during the selling season. Our model is an initial attempt to understand the effects of competition on pricing decisions of firms, and we make a few simplifying assumptions about the structure of the problem based on the previous results of Feng and Gallego (1995) and Gallego and van Ryzin (1994, 1997). First, although the magnitude of these mark-downs might be equally important, we focus only on the timing of the mark-downs. We assume that the initial price and the mark-down price are publicized and known in advance. The pre-specified prices may be a result of price lining practiced by some retailers or a result of an industry level consensus. We allow each company to make at most one price change. This restriction may be justified when the costs associated with price changes are considered. Moreover, Gallego and van Ryzin (1994) have shown that a single price change system is as effective as a more flexible pricing system, especially when the sales volume is high and price changes are costly.

Because of the complex structure of the stochastic solution for even the single firm case (Feng and Gallego 1995), and given our main purpose to study the effects of competition, we use deterministic demand rates in this work. Gallego and van Ryzin (1994, 1997) have shown that solutions to the deterministic problem may be used to construct asymptotically optimal heuristics for the stochastic problem.

Now, consider two competing firms,  $A$  and  $B$ . Assume that firm  $A$  has a

Table 1: Demand Rates for Two Competing Firms

		Firm $B$		
		$p_1$	$p_2$	stock-out
Firm $A$	$p_1$	$\lambda_1^A, \lambda_1^B$	$\lambda_F^A, \lambda_L^B$	$\lambda_{M1}^A, 0$
	$p_2$	$\lambda_L^A, \lambda_F^B$	$\lambda_2^A, \lambda_2^B$	$\lambda_{M2}^A, 0$
	stock-out	$0, \lambda_{M1}^B$	$0, \lambda_{M2}^B$	$0, 0$

starting inventory  $n^A$  and firm  $B$  has a starting inventory  $n^B$ . Both firms start with the same price  $p_1$  and each firm has the option to decrease the price to  $p_2$  at some time within a horizon of length  $t$ . The problem for company  $A$  and company  $B$  is to find the price switching times  $s^A$  and  $s^B$ , respectively, such that their profits over the entire selling season are optimized. We assume that  $p_1$  and  $p_2$  are fixed and known in advance.

We now give the representation of demand rates based on the prices charged by each firm (in the order of firm  $A$ , and firm  $B$ ) in Table 1. The demand structure in Table 1 models price completion. Firm  $A$  receives a demand rate of  $\lambda_1^A$  if both firms offer the high price  $p_1$  and a demand rate of  $\lambda_2^A$  if both firms offer the low price  $p_2$ . Similarly, firm  $B$  receives a demand rate of  $\lambda_1^B$  if both firms offer the high price, and a demand rate of  $\lambda_2^B$  if both firms offer the low price. Subscripts  $L$  and  $F$  are used to indicate the price leader and the price follower. If firm  $A$  has already lowered its price, while firm  $B$  is still charging the high price, firm  $A$  and firm  $B$  receive the demand rates  $\lambda_L^A$  and  $\lambda_F^B$ , respectively. If firm  $B$  is the leader in price change instead, firms  $A$  and  $B$  receive the demand rates  $\lambda_F^A$  and  $\lambda_L^B$ , respectively. When one firm  $B$  is out of stock, firm  $A$  observes a demand rate of  $\lambda_{M1}^A$  if the competitor is charging price  $p_1$ , and  $\lambda_{M2}^A$  if it is charging price  $p_2$ . Similarly, firm  $B$  observes the demand rates  $\lambda_{M1}^B$  and  $\lambda_{M2}^B$  when firm  $A$  is out of stock.

We first outline how these demand rates are determined in the context of the linear demand model. Then we describe our model and results for the case when demands are independent and each firm acts as a monopolist to optimize its own profits. Then, we present our results for the two competing firm (duopoly) case.

## 2.1 The Linear Demand Model

For the remainder of this paper, we assume that the two prices charged by firms impact each company's demand rate in a linear fashion. Linear demand systems have been widely used in the marketing and economics literature. In particular, we use the linear demand model in McGuire and Staelin (1983).

Assume that there are two firms, firm  $A$  and firm  $B$  that are offering two products (most likely, but not necessarily substitutable). If the prices are  $p^A$  and  $p^B$ , the demand rates for two products can be expressed by,

$$\lambda^A(p^A, p^B) = \mu S \left( 1 - \frac{\beta}{1-\theta} p^A + \frac{\beta\theta}{1-\theta} p^B \right), \quad (1)$$

$$\lambda^B(p^A, p^B) = (1-\mu) S \left( 1 + \frac{\beta\theta}{1-\theta} p^A - \frac{\beta}{1-\theta} p^B \right), \quad (2)$$

where  $0 \leq \mu \leq 1, 0 \leq \theta < 1$ , and  $\beta$  and  $S$  are positive constants. In this representation,  $S$  is a scale factor which equals the total demand rate when both prices are zero. The demand for both products increase linearly as  $S$  increase.  $\mu$  captures the absolute difference in demand, i.e., if both firms charge the same price, firms  $A$  and  $B$  get the  $100 \mu$  percent and  $100(1-\mu)$  percent of the total available demand, respectively.  $\theta$  is a measure of product substitutability. When  $\theta = 0$ , the demands are independent, and each firm behaves as a monopolist.  $\beta$  is a measure for the sensitivity of demand to price changes.

For this model, an arbitrary selection of parameters may result in negative demand rates. To guarantee non-negative demand rates, we set the following bounds on the prices

$$p^A \leq \frac{(1-\theta)}{\beta} + \theta p^B, \quad (3)$$

$$p^B \leq \frac{(1-\theta)}{\beta} + \theta p^A. \quad (4)$$

If a firm sets its price at its bound, the demand rate for that firm simply drops to zero, and the demand rate for the competitor reaches its maximum that is attainable with its current price.

To ensure that the total industry demand does not increase with an increase in either price, the parameter  $\mu$  for absolute difference in demand should be bounded as follows,

$$\frac{\theta}{1+\theta} \leq \mu \leq \frac{1}{1+\theta}. \quad (5)$$

## 2.2 The Case of Independent Demand: Optimal Mark-down Time for a Monopoly

Assume that the products offered by the two firms are not substitutable and thus their demands are independent. In our specific model, this means that the parameter for substitutability ( $\theta$ ) is zero. Using equations (1) and (2),

$$\begin{aligned} \lambda^A(p^A, p^B) &= \mu S (1 - \beta p^A) \\ \lambda^B(p^A, p^B) &= (1-\mu) S (1 - \beta p^B) \end{aligned}$$

As expected, when the products are not substitutable, the demand rate for one product is totally independent of the price charged for the other product. In particular, the demand rates in Table 1 are as follows for the independent demand case:

$$\begin{aligned}\lambda_1^A &= \lambda_F^A = \lambda_{M1}^A &= \mu S(1 - \beta p_1), \\ \lambda_1^B &= \lambda_F^B = \lambda_{M1}^B &= \mu(1 - S)(1 - \beta p_1), \\ \lambda_2^A &= \lambda_L^A = \lambda_{M2}^A &= \mu S(1 - \beta p_2), \\ \lambda_2^B &= \lambda_L^B = \lambda_{M2}^B &= \mu(1 - S)(1 - \beta p_2).\end{aligned}$$

Independency leads the firms to act as monopolists when making their pricing decisions.

Consider one of these firms, firm  $i$  which faces the problem of selling a fixed stock of  $n^i$  units in a given horizon of length  $t$ . Let the starting price be  $p_1$  which generates a demand rate of  $\lambda_1^i$ . The problem is to find the switching time  $s$  after which the price is  $p_2$  and the demand rate is  $\lambda_2^i$ . We assume that  $p_1$  and  $p_2$  are fixed and known in advance. As  $p_1 > p_2$ , the linear demand model ensures that  $\lambda_2^i > \lambda_1^i$ . Without loss of generality, we also assume that any unsold unit beyond time  $t$  is worthless. The objective of the firm is to maximize its total revenue from sales with price  $p_1$  and sales with price  $p_2$ , which is

$$\begin{aligned}R_M^i(s) &= p_1 \min\{n^i, N_1(0, s^i)\} + p_2 \min\{n^i - \min\{n^i, N_1(0, s^i)\}, N_2(s^i, t)\} \\ &= (p_1 - p_2) \min\{n^i, N_1(0, s^i)\} + p_2 \min\{n^i, N_1(0, s^i) + N_2(s^i, t)\},\end{aligned}$$

where  $N_j(a, b)$  is the units sold between time  $a$  and  $b$ ; we use subscript  $M$  to denote monopoly. For the deterministic demand case,  $N_j(a, b) = \lambda_j^i(b - a)$ ; hence the payoff function equals

$$R_M^i(s^i) = (p_1 - p_2) \min\{n^i, \lambda_1^i s^i\} + p_2 \min\{n^i, \lambda_1^i s^i + \lambda_2^i(t - s^i)\}.$$

Note that, in order for the firm to have an incentive to change its price, revenue rate should be increasing with the price change; i.e., we must have  $p_1 \lambda_1^i < p_2 \lambda_2^i$ . We also exclude the trivial case,  $n^i < \lambda_1^i t$ , for which the firm is able to finish off its inventory with the high original price.

**Lemma 1** *The optimal switching time for a single firm is given by,*

$$s_M^{i*} = \max\left\{0, \frac{n^i - \lambda_2^i t}{\lambda_1^i - \lambda_2^i}\right\}. \quad (6)$$

The solution is very simple. The monopoly firm will change the price at the latest possible time that it can still sell all of its inventory. Hence, the switching time is such that the demand exactly equals the total inventory. If the horizon is not

long enough to finish the inventory even with the low price ( $n^i > \lambda_2^i t$ ), the firm simply sets the price to  $p_2$  at the beginning of the selling season ( $s_M^{i*} = 0$ ). The firm's optimal revenue is,

$$\begin{aligned} R_M^{i*} &= p_1 \lambda_1^i s_M^{i*} + p_2 \lambda_2^i (t - s_M^{i*}) \\ &= p_1 \lambda_1^i s_M^{i*} + p_2 (n^i - \lambda_1^i s_M^{i*}). \end{aligned}$$

For a stochastic treatment of the problem where  $N_j(a, b)$  is a Poisson random variable with rate  $\lambda_j(b - a)$ , see Feng and Gallego (1995).

Note that these results are valid for any demand price relationship satisfying  $p_1 \lambda_1 < p_2 \lambda_2$ . Our linear demand model satisfies this condition if  $p_1 + p_2 > 1/\beta$  and specifically, we have the following specific switching times for firms  $A$  and  $B$ :

$$s_M^{A*} = \max\left\{0, \frac{n^A - \mu S(1 - \beta p_2)}{\mu S \beta (p_2 - p_1)}\right\}, \quad s_M^{B*} = \max\left\{0, \frac{n^B - (1 - \mu)S(1 - \beta p_2)}{(1 - \mu)S \beta (p_2 - p_1)}\right\}. \quad (7)$$

### 2.3 Case of Dependent Demand: Equilibrium Switching Times for a Duopoly

When the demands are not independent, the firms can no longer optimize their profits by only considering their own actions. We use a game theoretic model to find the equilibrium price switching times for the two firms. The game is non-cooperative with complete information. That is, the *rules* of the game are common knowledge. Each firm knows its own and its competitor's starting inventory levels, demand rates and payoff functions as well as the length of the selling season. With computerized and widely accessible reservation systems for the service industry and huge amounts of online data in the retailing industry, we believe that firms have the capability to have good estimates of their competitor's inventory levels, demand rates and motivations.

Using equations (1) and (2), the demand rates in Table 1 can be derived. We first derive the demand rates when both firms charge the same price.

$$\lambda_1^A = \mu S(1 - \beta p_1), \quad \lambda_2^A = \mu S(1 - \beta p_2), \quad (8)$$

$$\lambda_1^B = (1 - \mu)S(1 - \beta p_1), \quad \lambda_2^B = (1 - \mu)S(1 - \beta p_2), \quad (9)$$

Note that these demand rates are equal to the demand rates that these firms observe when the demands are totally independent.

Using equations (1–2) and (8–9), the demand rates when the firms charge different prices can be derived as follows:

$$\lambda_L^A = \frac{1}{(1 - \theta)} \lambda_2^A - \frac{\mu}{(1 - \mu)} \frac{\theta}{(1 - \theta)} \lambda_1^B,$$

$$\begin{aligned}
\lambda_F^A &= \frac{1}{(1-\theta)}\lambda_1^A - \frac{\mu}{(1-\mu)}\frac{\theta}{(1-\theta)}\lambda_2^B, \\
\lambda_L^B &= -\frac{(1-\mu)}{\mu}\frac{\theta}{(1-\theta)}\lambda_1^A + \frac{1}{(1-\theta)}\lambda_2^B, \\
\lambda_F^B &= -\frac{(1-\mu)}{\mu}\frac{\theta}{(1-\theta)}\lambda_2^A + \frac{1}{(1-\theta)}\lambda_1^B.
\end{aligned}$$

The demand rates for a firm when the its competitor is out of stock can be found by setting its competitor's price to its maximum prescribed by equations (3) and (4). Then,

$$\begin{aligned}
\lambda_{M1}^A &= \mu S(1+\theta)(1-\beta p_1) = (1+\theta)\lambda_1^A, \\
\lambda_{M2}^A &= \mu S(1+\theta)(1-\beta p_2) = (1+\theta)\lambda_2^A, \\
\lambda_{M1}^B &= \mu(1-S)(1+\theta)(1-\beta p_1) = (1+\theta)\lambda_1^B, \\
\lambda_{M2}^B &= \mu(1-S)(1+\theta)(1-\beta p_2) = (1+\theta)\lambda_2^B.
\end{aligned}$$

Having described the demand rates in terms of  $\lambda_1^A, \lambda_2^A, \lambda_1^B$  and  $\lambda_2^B$ , we observe the following,

$$\frac{\lambda_1^A}{\lambda_1^B} = \frac{\lambda_2^A}{\lambda_2^B} = \frac{\lambda_L^A}{\lambda_L^B} = \frac{\lambda_F^A}{\lambda_F^B} = \frac{\lambda_{M1}^A}{\lambda_{M1}^B} = \frac{\lambda_{M2}^A}{\lambda_{M2}^B} = \frac{\mu}{(1-\mu)} \quad (10)$$

In other words, for all pairs of states (prices charged or inventory positions), firm  $A$ 's demand rate is always the same multiple of the demand rate that firm  $B$  would get if the states were reversed. This helps us to rescale the problem such that the firms  $A$  and  $B$  are identical except for their starting inventory levels. In order to do so, we first rescale the starting inventory positions so that,

$$m^A = \frac{n^A}{S\mu}, \quad m^B = \frac{n^B}{S(1-\mu)}. \quad (11)$$

While  $n^A$  and  $n^B$  define the absolute starting inventory levels for firm  $A$  and  $B$ ,  $m^A$  and  $m^B$  measure firm  $A$ 's and firm  $B$ 's starting inventory positions relative to their market potential (Remember that  $S$  is the total demand rate if both firms charge zero price). We also rescale the demand rates for firm  $A$  and firm  $B$  with  $S\mu$  and  $S(1-\mu)$ , respectively, so that we have the same demand rates for both firms. Namely,  $\lambda_1, \lambda_2, \lambda_L, \lambda_F, \lambda_{M1}$  and  $\lambda_{M2}$ .

Note also that the following properties hold for the linear demand model:

$$\lambda_L + \lambda_F = \lambda_2 + \lambda_1, \quad (12)$$

$$\lambda_L - \lambda_F = (\lambda_2 - \lambda_1)\frac{(1+\theta)}{(1-\theta)} = \beta(p_2 - p_1)\frac{(1+\theta)}{(1-\theta)}. \quad (13)$$

Equation (12) states that the sum of the demand rates for the leader and the follower in the price change is equal to the demand rates that these firms would

get if they were completely independent. Equation (13) states that the difference in leader's and follower's demand rates increases as the magnitude of price change and the rate of substitution (i.e., demand correlation) gets substantial. When the demand for two products are completely independent (i.e.,  $\theta = 0$ ),  $\lambda_2 = \lambda_L$  and  $\lambda_1 = \lambda_F$ .

We assume that both firms are over-stocked with respect to the demand with the initial price (i.e.,  $m^A > \lambda_1 t$  and  $m^B > \lambda_1 t$ ), for otherwise, the problem is trivial and both firms will charge  $p_1$  throughout the selling season. Without loss of generality, assume that  $m^A \geq m^B$ , i.e., the firm  $A$  is over-stocked relative to the firm  $B$ . We are now ready to present our main result. We first describe the equilibrium switching times and then go on to formally state and prove that these switching times are unique Nash equilibrium.

We show that, in equilibrium, firm  $A$  (being more over-stocked) lowers its price earlier than firm  $B$  does, and price switching times  $(s^{A*}, s^{B*})$  satisfy the following conditions:

$$\lambda_1 s^{A*} + \lambda_L (s^{B*} - s^{A*}) + \lambda_2 (t - s^{B*}) = m^A, \quad (14)$$

$$\lambda_1 s^{A*} + \lambda_F (s^{B*} - s^{A*}) + \lambda_2 (t - s^{B*}) = m^B. \quad (15)$$

The solution to (14) and (15) indicates that, in equilibrium, each firm switches its price at a point such that it runs out of stock right at the end of the horizon. Closed form solution to (14) and (15) is the following,

$$s^{A*} = \frac{\lambda_2 (\lambda_L - \lambda_F) t - (\lambda_L - \lambda_2) m^B - (\lambda_2 - \lambda_F) m^A}{(\lambda_2 - \lambda_1) (\lambda_L - \lambda_F)}, \quad (16)$$

$$s^{B*} = \frac{\lambda_2 (\lambda_L - \lambda_F) t - (\lambda_L - \lambda_1) m^B - (\lambda_1 - \lambda_F) m^A}{(\lambda_2 - \lambda_1) (\lambda_L - \lambda_F)}. \quad (17)$$

**Theorem 1**  $(s^{A*}, s^{B*})$  is a Nash equilibrium, if  $p_1 + p_2 > 1/\beta$ .

**Proof:** See Appendix B.1.

**Theorem 2**  $(s^{A*}, s^{B*})$  is unique, if  $p_1 + p_2 > 1/\beta$ .

**Proof:** See Appendix B.2.

The only condition for the unique Nash equilibrium ( $p_1 + p_2 > 1/\beta$ ) states that the revenue rate should be increasing (i.e.,  $p_2 \lambda_2 > p_1 \lambda_1$ ) with the mark-down even if both firms are totally independent of each other.

We now study the qualitative properties of the equilibrium. We first observe that the relatively over-stocked firm switches its price before its competitor does.

Without competition, this is quite intuitive since the more over-stocked firm will need more time during which it faces a higher demand rate to liquidate its inventory and thus will lower its price earlier. Our results show that, even in the existence of demand interactions, the more over-stocked firm still lowers its price earlier than its competitor does.

From (16) and (17), the length of the period during which the firms charge different prices equals,

$$s^{B*} - s^{A*} = \frac{m^A - m^B}{\lambda_L - \lambda_F} = \frac{(m^A - m^B)(1 - \theta)}{(p_1 - p_2)\beta(1 + \theta)}. \quad (18)$$

Note that the formula describes two forces in action. While the difference in inventory positions ( $m^A - m^B$ ) compels the firms to act individually, the intensity of demand interaction ( $\lambda_L - \lambda_F$  or  $\theta$ ) forces the firms to act together in their price switching decisions. Note again that  $\theta$  is a measure of demand dependency. As the dependency increases ( $\theta$  increases), firms tend to follow each other's decisions, and switch almost simultaneously when  $\theta$  gets close to 1. If there is no interaction ( $\theta = 0$ ), the firms switch at monopolistic switching times. These two forces are simplifications of two general phenomena. While the differences in the structures and operations of different firms lead the firms to follow different strategies, the product substitutability and hence competition limits them in doing so.

At the other extreme, the products are not substitutable, and hence the demand rates of the firms depend on their own prices only. As described in section 2.2, when  $\theta = 0$ , firm  $A$  (or  $B$ ) would observe the demand rate  $\lambda_1$  (or  $\lambda_2$ ) with price  $p_1$  (or  $p_2$ ), regardless of its competitor's price. When this is the case, the firms command their own markets to act like monopolists and switch their prices at the monopoly switching times described by Lemma 1. From equation (7), firm  $A$  changes its price at time  $s_M^{A*} = \frac{m^A - \lambda_1 t}{\lambda_2 - \lambda_1}$ , and firm  $B$  changes its price at time  $s_M^{B*} = \frac{m^B - \lambda_1 t}{\lambda_2 - \lambda_1}$ . From the discussion above, with demand interaction, it is clear that the length of the time period during which firms charge different prices is smaller, relative to the case of no demand interaction. Of particular interest is the switching time of each firm in the existence of demand interaction compared to its switching time when the demands are independent. The next lemma makes this comparison.

**Lemma 2** *The more over-stocked firm changes the price later, and the less over-stocked firm changes the price earlier, than they would change if there was no demand interaction (and hence, no competition).*

**Proof:** If we replace  $m^B$  in equation (16) by  $m^A$ , we obtain  $s_M^{A*}$ . Thus,  $s^{A*} \geq s_M^{A*}$ . If we replace  $m^A$  in equation (17) by  $m^B$ , we obtain  $s_M^{B*}$ . Thus,  $s^{B*} \leq s_M^{B*}$ .

Intuitively, the more over-stocked firm will mark its price down first, and hence, during the time it charges a lower price than its competitor, will enjoy a higher demand rate than it would if there was no demand interaction. Therefore it needs less time to deplete its inventory and can use more time to sell at a higher price, and hence should change the price later. On the other hand, the less over-stocked firm will mark its price down second, and hence, during the time it charges a higher price than its competitor's, it will face a demand rate lower than it would face if there was no demand interaction. Therefore, it can sell fewer units than it would normally sell during the time it charges the high price, and so should change the price earlier.

Now, we compare the revenue of each firm under competition to its revenue when there is no demand interaction (and hence, no competition).

**Lemma 3** *With demand interaction, the more over-stocked firm has a higher payoff and the less over-stocked firm has a lower payoff (both compared to the payoffs when there is no demand interaction). The total industry payoff increases with demand interaction (compared to the total payoff with no demand interaction) if and only if  $\mu\lambda_1 > (1 - \mu)\lambda_2$ .*

**Proof:** Let  $R_M^{A*}$  and  $R_M^{B*}$  be the optimal profits for firms  $A$  and  $B$ , respectively, if there is no demand interaction. Also let  $R_A^*$  and  $R_B^*$  be the optimal profits for firm  $A$  and firm  $B$ , respectively, with demand interaction. Then,

$$\begin{aligned} R_M^{A*} &= \mu S[p_1\lambda_1 s_M^{A*} + p_2\lambda_2(t - s_M^{A*})], \\ R_M^{B*} &= (1 - \mu)S[p_1\lambda_1 s_M^{B*} + p_2\lambda_2(t - s_M^{B*})], \\ R^{A*} &= \mu S[p_1\lambda_1 s^{A*} + p_2[\lambda_L(s^{B*} - s^{A*})] + \lambda_2(t - s^{B*})], \\ R^{B*} &= (1 - \mu)S[p_1\lambda_1 s^{A*} + p_1[\lambda_F(s^{B*} - s^{A*}) + \lambda_2(t - s^{B*})]]. \end{aligned}$$

Let

$$\delta \equiv \frac{(m^A - m^B)}{(\lambda_2 - \lambda_1)(\lambda_L - \lambda_F)}.$$

It can be shown that

$$s^{A*} - s_M^{A*} = \delta(\lambda_L - \lambda_2), \quad s_M^{B*} - s^{B*} = \delta(\lambda_1 - \lambda_F),$$

which leads to

$$\begin{aligned} R^{A*} - R_M^{A*} &= \mu S(p_1 - p_2)\delta\lambda_1(\lambda_L - \lambda_2), \\ R^{B*} - R_M^{B*} &= (1 - \mu)S(p_2 - p_1)\delta\lambda_2(\lambda_1 - \lambda_F), \\ R^{A*} + R^{B*} - R_M^{A*} - R_M^{B*} &= S(p_1 - p_2)\delta[\mu\lambda_1(\lambda_L - \lambda_2) - (1 - \mu)\lambda_2(\lambda_1 - \lambda_F)] \end{aligned}$$

Define

$$\psi = \frac{S(m^A - m^B)(p_1 - p_2)\theta}{(\lambda_2 - \lambda_1)(1 + \theta)} = \frac{S(m^A - m^B)\theta}{\beta(1 + \theta)}.$$

One can show that,

$$\lambda_L - \lambda_2 = \lambda_1 - \lambda_F = \frac{\theta}{1 - \theta}(\lambda_2 - \lambda_1),$$

which leads to,

$$\begin{aligned} R^{A*} - R_M^{A*} &= \psi\mu\lambda_1 \\ R^{B*} - R_M^{B*} &= \psi(\mu - 1)\lambda_2 \\ R^{A*} + R^{B*} - R_M^{A*} - R_M^{B*} &= \psi[\mu\lambda_1 - (1 - \mu)\lambda_2] \end{aligned}$$

The results follow.

For the linear model, as the demand interaction becomes more intense ( $\theta$  gets closer to 1), the payoff for the more over-stocked firm increases, the payoff for the less over-stocked firm decreases. This result may have an advertising implication: to increase its payoffs, the more over-stocked firm should expend in informing consumers in the industry about its price and emphasize the similarity of its product to other products offered by competing firms. The less over-stocked firm, on the other hand, should emphasize the distinctive properties of its product. Remember that  $\mu$  stands for the market share of the more over-stocked firm. The total payoff increases if only if  $\mu\lambda_1 - (1 - \mu)\lambda_2 > 0$ , or  $\mu > \lambda_2/(\lambda_1 + \lambda_2)$ . That is, the competition helps the industry to improve its revenues, if and only if the more over-stocked firm is sufficiently large, as this firm is the only one that benefits from the competition.

We note that the above analysis does not compare the payoffs of two firms when they compete to their payoffs when they cooperate. Rather, we compare their payoffs with a specified level of demand interaction to their payoffs when there is no demand interaction.

### An Example

Assume that  $S = 70$  and  $\mu = 0.4$ . That is, firm  $A$  and firm  $B$  have the market shares 40% and 60%, respectively, and if both firms charge zero price, the maximum total demand rate for the two firms is 70 per day. Assume that firm  $A$  has a stock of  $n^A = 1,280$  units and firm  $B$  has a stock of  $n^B = 1,440$  units to be sold in a season of 100 days. After rescaling the starting stock levels with the market share ( $\mu$  and  $1 - \mu$ ) and the total maximum demand rate ( $S$ ), we obtain  $m^A = 320/7$  and  $m^B = 240/7$ . Note that although the firm  $A$  has lower

starting inventory, it is more over-stocked relative to the firm  $B$ . Let  $\beta = 1/14$  and for now assume that the demands are independent, i.e.,  $\theta = 0$ . Assume also that  $p_1 = 10$  and  $p_2 = 6$ . In this case,  $\lambda_1 = 2/7$  and  $\lambda_2 = 4/7$ . As the products are not substitutable, each firm acts as a monopolist: firm  $A$  switches its price on day 40 and firm  $B$  switches its price on day 80. At the end, the payoffs for firm  $A$  and firm  $B$  are \$8,960 and \$12,480, respectively, with a total industry payoff of \$21,440.

Now assume that the demands are dependent such that  $\theta = 1/3$ . Then  $\lambda_F = 1/7$ ,  $\lambda_L = 5/7$  and  $\lambda_{M1} = 8/21$ ,  $\lambda_{M2} = 16/21$ . In equilibrium, firm  $A$  changes the price on day 50 and firm  $B$  changes the price on day 70. Being more over-stocked, firm  $A$  achieves its maximum payoff of \$9,280, while firm  $B$  has to react to firm  $A$  and can only get a payoff of \$11,520. Demand interaction helped firm  $A$  increase its payoff while firm  $B$ 's payoff decreased with the demand interaction. Note that firm  $A$  changes its price 10 days later, and firm  $B$  changes its price 10 days earlier than they would if there was no demand interaction. With competition, the time period during which the firms charge differential prices is only 20 days. The total payoff has also decreased to \$20,800.

### 3 The Mark-up Problem

We develop a similar model when the firms actually mark the prices up. This kind of price adjustment can be observed in yield management for airlines and hotels. The assumptions of the mark-up model are exactly the same as those made for the mark-down model in Section 2. We first describe our model for the single firm (monopoly) case. Then, we present our model for the two firm (duopoly) case. Finally, we discuss the discount fare allocation problem, and show how the results of our mark-up model can be used via a transformation. In this section, we assume that the starting inventory levels and demand rates are already rescaled by each market share and thus we can treat the firms act as if they are symmetric except for their starting inventory levels.

#### 3.1 Optimal Mark-up Time for a Monopoly

Consider the same setting as in Section 2.2. We now assume  $p_2 > p_1$  and thus  $\lambda_2^i < \lambda_1^i$ . Note that we now have the restriction  $p_2\lambda_2^i < p_1\lambda_1^i$  or  $p_1 + p_2 < 1/\beta$ , since otherwise the firm would mark the price up at time 0. We also exclude the trivial case,  $\lambda_1^i t < n^i$ , in which the firm is not able to sell off all inventory even with the low price. The optimal time to switch is,

$$s_M^{i*} = \min\left\{t, \frac{n^i - \lambda_2^i t}{\lambda_1^i - \lambda_2^i}\right\}.$$

The equation states that if the demand rate with the low price is not large enough to deplete all the inventory, the firm would not switch the price at all. The firm now switches the price at the earliest (as opposed to the latest in Section 2.2) time that will liquidate all of its inventory.

### 3.2 Equilibrium Mark-up Times for Duopoly

We use a model similar to the one described in Section 2.3. We now have  $p_2 > p_1$  and  $p_1\lambda_1 > p_2\lambda_2$ . Also assume now that  $\lambda_1 t \geq m^A \geq m^B$ .

The equilibrium is similar to that of the mark-down problem. Again, each firm depletes its inventory right at the end of its selling season. But, this time the less over-stocked firm (firm  $B$ ) switches its price first at time

$$s^{B*} = \frac{\lambda_2(\lambda_L - \lambda_F)t - (\lambda_L - \lambda_2)m^A - (\lambda_2 - \lambda_F)m^B}{(\lambda_2 - \lambda_1)(\lambda_L - \lambda_F)}, \quad (19)$$

followed by the firm with a higher inventory (firm  $A$ ) switching at time

$$s^{A*} = \frac{\lambda_2(\lambda_L - \lambda_F)t - (\lambda_L - \lambda_1)m^A - (\lambda_1 - \lambda_F)m^B}{(\lambda_2 - \lambda_1)(\lambda_L - \lambda_F)}. \quad (20)$$

The solution is exactly the same as the solution for the mark-down problem except that the  $m^A$  and  $m^B$  are interchanged in the formulae. Now, the only condition under which  $(s^{A*}, s^{B*})$  is a unique Nash equilibrium is  $p_1\lambda_1 > p_2\lambda_2$  or  $p_1 + p_2 < 1/\beta$ .

The difference in switching times,  $s^{A*} - s^{B*}$ , is now equal to  $\frac{m^A - m^B}{\lambda_F - \lambda_L}$ . The results of the mark-down problem can easily be translated into the mark-up problem. The less over-stocked firm marks its price up later, and the more over-stocked firm marks its price up earlier, than they would without demand interaction. Thus, the time period for which the firms charge different prices is shorter when demand interaction or competition is present. Also, with demand interaction, the profit of the more over-stocked firm increases, while profit of the less over-stocked firm decreases.

We now consider the same example discussed earlier in Section 2, only with reversed demand and price parameters. That is,  $p_1 = \$6$ ,  $p_2 = \$10$ ,  $\lambda_1 = 4/7$ ,  $\lambda_2 = 2/7$ ,  $\lambda_L = 1/7$  and  $\lambda_F = 5/7$ . We still have  $n^A = 1,280$ ,  $n^B = 1,440$  and  $t = 100$ . The equilibrium is a time-reversed version of the equilibrium in the mark-down problem. That is, firm  $B$  switches at time  $s^{A*} = 30 (= 100 - 70)$  and firm  $A$  switches at time  $s^{B*} = 50 (= 100 - 50)$ .

### 3.3 Discount Fare Allocation Problem

Models similar in spirit can be used to model competition in allocation of seats to regular and discount customers for service companies.

## Optimal Closing Time of Discount Fares for a Monopoly

Consider a firm which faces the problem of selling a fixed stock of  $n$  units (airline seats) to two different type of customers (regular and discount) with different prices. The regular demand over the entire horizon is less than  $n$ , while the regular and discount demand combined is more than  $n$ . The objective of the firm should be to meet as much regular demand as possible, while also being able to sell all its stock. The allocation can be done by closing the discount fare at some point in the selling horizon, leaving all remaining stock to the remaining regular demand. To formalize the model, again let  $t$  be the length of the selling horizon,  $p_D$  be the price for the discount tickets,  $p_R$  be the price for the regular tickets. Let  $\lambda_{R1}$  and  $\lambda_{D1}$  be the demand rates for regular and discount customers, respectively, before closing the discount fare; let  $\lambda_{R2}$  be the regular demand rate after closing the discount fare. Note that  $\lambda_{R2}$  may be greater than  $\lambda_{R1}$ , because of *diversion*, or upgrading of originally discount customers to regular fare.

For the deterministic problem, the closing time of the discount class should be such that the firm sells the maximum number of regular tickets while also depletes all of its stock. Thus, the optimal closing time  $s_M^*$  for a monopoly is such that

$$(\lambda_{R1} + \lambda_{D1})s_M^* + \lambda_{R2}(t - s_M^*) = n,$$

which gives

$$s_M^* = \frac{n - \lambda_{R2}t}{\lambda_{D1} + \lambda_{R1} - \lambda_{R2}}.$$

If the total demand is less than the capacity ( $(\lambda_{D1} + \lambda_{R1})t < n$ ), it is optimal not to close the discount class at all.

The solution can be also obtained by transforming the fare allocation problem into a mark-up problem, by using average price and aggregate demand parameters. To find the equilibrium discount fare closing times, the following transformation can be used:

$$\begin{aligned} \lambda_1 &= \lambda_{D1} + \lambda_{R1}, \\ \lambda_2 &= \lambda_{R2}, \\ p_1 &= (p_D\lambda_{D1} + p_R\lambda_{R1})/(\lambda_{D1} + \lambda_{R1}), \\ p_2 &= p_R. \end{aligned}$$

## Equilibrium Closing Times of Discount Fares for a Duopoly

Assume that firm  $A$  and firm  $B$  have starting inventories of  $n^A$  and  $n^B$ , respectively. Discount demand rates for both firms are:

		Firm $B$ 's discount fare	
		open	close
Firm $A$ 's discount fare	open	$\lambda_{D1}, \lambda_{D1}$	$\lambda_{DF}, 0$
	close	$0, \lambda_{DF}$	$0, 0$

Similarly, regular demand rates are:

		Firm $B$ 's discount fare	
		open	close
Firm $A$ 's discount fare	open	$\lambda_{R1}, \lambda_{R1}$	$\lambda_{RF}, \lambda_{RL}$
	close	$\lambda_{RL}, \lambda_{RF}$	$\lambda_{R2}, \lambda_{R2}$

We did not identify the demand rates for each firm when its competitor is sold out, since these demand rates do not affect the equilibrium switching times as long as they are within reasonable bounds described in Theorem 2.

Once again, the problem can be converted into a competitive mark-up problem by using the following transformation:

$$\begin{aligned}\lambda_1 &= \lambda_{R1} + \lambda_{D1}, \\ \lambda_2 &= \lambda_{R2}, \\ \lambda_L &= \lambda_{RL}, \\ \lambda_F &= \lambda_{RF} + \lambda_{DF}.\end{aligned}$$

However, we now will have four average prices: one for the case when both firms have their discount classes open, one for the leader, one for the follower and one for the case when both firms have their discount classes closed. But, we have seen in Section 3.2 that the analysis does not depend on the prices as long as the demands are deterministic and the assumptions still hold ( $p_1\lambda_1 > p_2\lambda_2$ , and  $\lambda_2 t \leq n^B \leq n^A t$ ).

With this transformation, the results of Section 3.2 can be extended to the discount fare allocation problem. Our model suggests that, with competition, the firm with a smaller capacity closes its discount class first. With increased demand interaction, the firm with a smaller capacity would close its discount class later and the firm with a larger capacity would close its discount earlier, which results in a shorter period of time during which the competing firms have different levels of fare availability.

## 4 Conclusion

In this section, we have studied the effects of competition on firms in a market with perfect information. The two competing firms face symmetric demand rates; the only differentiation between the two firms is their initial inventories or capacities.

We have used the results of previous researchers on the single-firm problem to simplify the structure of the two-firm problem. Using a game theoretic model, we show that a unique Nash equilibrium exists under mild conditions, and we derive the equilibrium price switching times.

A major result of our model is that a competing firm will choose to change its price at a time within the selling season so that it will just sell off all its inventory. In the mark-down problem, the more over-stocked firm will act as a price leader by lowering its price first, whereas in the mark-up problem the less over-stocked firm will raise its price first. We have shown that, while differences in the starting inventory positions compel the competing firms to switch their prices at different times in the selling season, an increased degree of demand interaction forces them to follow each other's actions closely. The length of time for the two firms to charge different prices is shown to be proportional to the difference in initial inventory levels but inversely proportional to the difference in demand rates under the two prices. That is, the stocking difference between the two firms as measured by initial inventories or capacities tends to increase the gap in price switching times, while the demand differential under the two potentially different prices tends to force the competing firms to follow each other's pricing decisions. As a result of the competition, the more over-stocked firm changes its price later than it would without competition, and the less over-stocked firm changes its price earlier than it would without competition. The behavior predicted by our model is not difficult to observe in practice.

Another major result is that the more over-stocked firm benefits from competition while the less over-stocked firm stands to lose from increased competition. This result might not be surprising if we note that in the mark-down problem the more over-stocked firm lowers its price later than it would when there is no competition, and thus sells more units with the high price. Hence, competition benefits the more over-stocked firm. On the other hand, the less over-stocked firm lowers its price earlier than it would when there is no competition, and thus sells fewer units with the high price. Hence, competition hurts the less over-stocked firm. A similar argument applies for the mark-up problem.

There are several directions for future research. First, we may allow for multiple price changes. Moreover, the price levels may be decision variables. Introducing uncertainty in demand rates or incomplete information available to the parties of the game may further strengthen our results. However, all these extensions will complicate the game theoretic analysis, which may make it harder to identify and interpret the equilibrium.

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# Appendix

## A Proof of Theorem 1

Let  $R^A(s^A, s^B)$  be the payoff for firm  $A$  if it switches its price at time  $s^A$  and if the firm  $B$  switches the price at time  $s^B$ . Likewise, let  $R^B(s^A, s^B)$  be the payoff for firm  $B$  for the switching times  $s^A$  and  $s^B$ . We divide the lengthy proof into several parts.

**Part I:** We have to show,

$$R^A(s^{A*}, s^{B*}) \geq R^A(s^A, s^{B*}), \quad \text{for } 0 \leq s^A \leq t.$$

**Part I.1:**

$$R^A(s^{A*}, s^{B*}) \geq R^A(s^A, s^{B*}), \quad \text{for } 0 \leq s^A \leq s^{A*}.$$

Note that

$$\begin{aligned} R^A(s^{A*}, s^{B*}) &= p_1 \lambda_1 s^{A*} + p_2 (m^A - \lambda_1 s^{A*}), \\ R^A(s^A, s^{B*}) &= p_1 \lambda_1 s^A + p_2 (m^A - \lambda_1 s^A), \end{aligned}$$

since for both cases when  $s^A = s^{A*}$  and when  $s^A \leq s^{A*}$ , firm  $A$  is able to deplete all of its inventory. Then,

$$R^A(s^{A*}, s^{B*}) - R^A(s^A, s^{B*}) = (p_1 - p_2) \lambda_1 (s^{A*} - s^A),$$

which is always positive.

**Part I.2:**

$$R^A(s^{A*}, s^{B*}) \geq R^A(s^A, s^{B*}), \quad \text{for } s^{A*} \leq s^A \leq s^{B*}.$$

In this case, firm  $B$  will be stocked out, say at time  $\ell$ , before the end of the horizon. Then,

$$\lambda_1 s^A + \lambda_F (s^{B*} - s^A) + \lambda_2 (\ell - s^{B*}) = m^B. \quad (21)$$

Recall that we already have

$$\lambda_1 s^{A*} + \lambda_F (s^{B*} - s^{A*}) + \lambda_2 (t - s^{B*}) = m^B. \quad (22)$$

At this point, we need to give the conditions under which firm  $A$  is not able to deplete all of its inventory by postponing its switching time. Otherwise, firm  $A$  can definitely increase its payoff thus  $(s^{A*}, s^{B*})$  is no longer an equilibrium. Let us find the minimum  $\lambda_{M2}$  at which firm  $A$  can also deplete its inventory before

the end of the horizon. We should have a solution for the minimum  $\lambda_{M2}$  from the following equation:

$$\lambda_1 s + \lambda_L(s^{B^*} - s^A) + \lambda_2(\ell - s^{B^*}) + \lambda_{M2}(t - \ell) = m^A. \quad (23)$$

Using (21) and (23),

$$(\lambda_L - \lambda_F)(s^{B^*} - s^A) + \lambda_{M2}(t - \ell) = m^A - m^B.$$

Noting that  $(\lambda_L - \lambda_F)(s^{A^*} - s^{B^*}) = m^A - m^B$ , we have

$$(\lambda_L - \lambda_F)(s^{A^*} - s^A) + \lambda_{M2}(t - \ell) = 0.$$

Using (21) and (22), we have

$$(\lambda_F - \lambda_1)(s^A - s^{A^*}) + \lambda_2(t - \ell) = 0. \quad (24)$$

And we have,

$$\lambda_{M2} = \lambda_2 + \lambda_2 \frac{\lambda_L - \lambda_1}{\lambda_1 - \lambda_F}. \quad (25)$$

That is, firm  $A$  will be able to deplete all its inventory at time  $t$  only if  $\lambda_{M2}$  is given by equation (25). For

$$\lambda_{M2} \leq \lambda_2 \left( 1 + \frac{\lambda_L - \lambda_1}{\lambda_1 - \lambda_F} \right), \quad (26)$$

firm  $A$  is not able to finish its inventory, if its switching time exceeds  $s^{A^*}$ .

Now assuming that equation (26) is satisfied, we compare the payoff when  $s^A = s^{A^*}$  and the payoff when  $s^{A^*} \leq s^A \leq s^{B^*}$ .

$$\begin{aligned} R^A(s^{A^*}, s^{B^*}) &= p_1 \lambda_1 s^{A^*} + p_2 [\lambda_L (s^{B^*} - s^{A^*}) + \lambda_2 (t - s^{B^*})], \\ R^A(s^A, s^{B^*}) &= p_1 \lambda_1 s^A + p_2 [\lambda_L (s^{B^*} - s^A) + \lambda_2 (\ell - s^{B^*}) + \lambda_{M2} (t - \ell)], \\ R^A(s^{A^*}, s^{B^*}) - R^A(s^A, s^{B^*}) &= (p_1 \lambda_1 - p_2 \lambda_L) (s^{A^*} - s^A) + p_2 (\lambda_{M2} - \lambda_2) (\ell - t). \end{aligned}$$

Using equation (24),

$$R^A(s^{A^*}, s^{B^*}) - R^A(s^A, s^{B^*}) = (s^{A^*} - s^A) \left[ (p_1 \lambda_1 - p_2 \lambda_L) + p_2 (\lambda_{M2} - \lambda_2) \frac{(\lambda_1 - \lambda_F)}{\lambda_2} \right],$$

which is positive only if

$$\lambda_{M2} \leq \lambda_2 \left[ 1 + \frac{p_2 \lambda_L - p_1 \lambda_1}{p_2 (\lambda_1 - \lambda_F)} \right]. \quad (27)$$

**Part I.3:**

$$R^A(s^{A^*}, s^{B^*}) \geq R^A(s^A, s^{B^*}), \quad \text{for } s^{B^*} \leq s^A \leq t.$$

Assume  $s^A$  is such that firm  $B$  still has some inventory, when firm  $A$  changes the price. Then,

$$\lambda_1 s^{B*} + \lambda_L (s^A - s^{B*}) + \lambda_2 (\ell - s^A) = m^B.$$

Also using

$$\lambda_1 s^{B*} + \lambda_F (s^{B*} - s^{A*}) + \lambda_2 (t - s^{B*}) = m^B,$$

We have,

$$(\lambda_1 - \lambda_F)(s^{B*} - s^{A*}) + (\lambda_L - \lambda_2)(s^A - s^{B*}) - \lambda_2 (t - \ell) = 0. \quad (28)$$

On the other hand,

$$R^A(s^A, s^{B*}) = p_1[\lambda_1 s^{B*} + \lambda_F (s^A - s^{B*})] + p_2[\lambda_2 (\ell - s^A) + \lambda_{M2}(t - \ell)],$$

$$\begin{aligned} R^A(s^{A*}, s^{B*}) - R^A(s^A, s^{B*}) &= (p_2 \lambda_L - p_1 \lambda_1)(s^{B*} - s^{A*}) \\ &+ (p_2 \lambda_2 - p_1 \lambda_F)(s^A - s^{B*}) \\ &- (p_2 \lambda_{M2} - p_2 \lambda_2)(t - \ell). \end{aligned}$$

Using equation (28),

$$\begin{aligned} R^A(s^{A*}, s^{B*}) - R^A(s^A, s^{B*}) &= \\ &\left[ (p_2 \lambda_L - p_1 \lambda_1) - \frac{(p_2 \lambda_{M2} - p_2 \lambda_2)(\lambda_1 - \lambda_F)}{\lambda_2} \right] (s^{B*} - s^{A*}) \\ &+ \left[ (p_2 \lambda_2 - p_1 \lambda_F) - \frac{(p_2 \lambda_{M2} - p_2 \lambda_2)(\lambda_L - \lambda_2)}{\lambda_2} \right] (s^A - s^{B*}), \end{aligned}$$

which is positive only if

$$\lambda_{M2} \leq \lambda_2 \left[ 1 + \frac{(p_2 \lambda_L - p_1 \lambda_1)}{p_2 (\lambda_1 - \lambda_F)} \right], \quad (29)$$

and

$$\lambda_{M2} \leq \lambda_2 \left[ 1 + \frac{(p_2 \lambda_2 - p_1 \lambda_F)}{p_2 (\lambda_L - \lambda_2)} \right]. \quad (30)$$

#### Part I.4:

$$R^A(s^{A*}, s^{B*}) \geq R^A(s^A, s^{B*}), \quad \text{for } s^{B*} \leq s^A \leq t.$$

Assume that firm  $B$ 's inventory is depleted before firm  $A$  changes the price. From Part I.3, we know that  $R^A(s^{A*}, s^{B*}) \geq R^A(s^A, s^{B*})$  for  $s^A = \ell$ . Now assume  $s^A > \ell$ . Then,

$$R^A(s^A, s^{B*}) = p_1[\lambda_1 s^{B*} + \lambda_F (\ell - s^{B*}) + \lambda_{M1}(s^A - \ell)] + p_2 \lambda_{M2}(t - s^A)$$

But,

$$R^A(\ell, s^{B*}) = p_1[\lambda_1 s^{B*} + \lambda_F (\ell - s^{B*})] + p_2 \lambda_{M2}(t - \ell),$$

$$R^A(\ell, s^{B*}) - R^A(s^A, s^{B*}) = (p_2\lambda_{M2} - p_1\lambda_{M1})(\ell - s^A).$$

For

$$p_2\lambda_{M2} - p_1\lambda_{M1} \geq 0, \quad (31)$$

$$R^A(\ell, s^{B*}) - R^A(s^A, s^{B*}) \geq 0 \text{ and } R^A(s^{A*}, s^{B*}) \geq R^A(s^A, s^{B*}).$$

**Part II:** We need to show,

$$R^B(s^{A*}, s^{B*}) \geq R^B(s^{A*}, s^B), \quad \text{for } 0 \leq s^B \leq t.$$

**Part I.1:**

$$\begin{aligned} R^B(s^{A*}, s^{B*}) &\geq R^B(s^{A*}, s^B), \quad \text{for } 0 \leq s^B \leq s^{A*} \\ R^B(s^{A*}, s^B) &= p_1\lambda_1 s^B + p_2(m^B - \lambda_1 s^B), \\ R^B(s^{A*}, s^{B*}) &= p_1[\lambda_1 s^{A*} + \lambda_F(s^{B*} - s^{A*})] + p_2[m^B - \lambda_1 s^{A*} - \lambda_F(s^{B*} - s^{A*})], \\ R^B(s^{A*}, s^{B*}) - R^B(s^{A*}, s^B) &= (p_1 - p_2)[\lambda_1(s^{A*} - s^B) + \lambda_F(s^{B*} - s^{A*})] \geq 0. \end{aligned}$$

**Part II.2:**

$$\begin{aligned} R^B(s^{A*}, s^{B*}) &\geq R^B(s^{A*}, s^B), \quad \text{for } s^{A*} \leq s^B \leq s^{B*} \\ R^B(s^{A*}, s^B) &= p_1[\lambda_1 s^{A*} + \lambda_F(s^B - s^{A*})] + p_2[m^B - \lambda_1 s^{A*} - \lambda_F(s^B - s^{A*})], \\ R^B(s^{A*}, s^{B*}) &= p_1[\lambda_1 s^{A*} + \lambda_F(s^{B*} - s^{A*})] + p_2[m^B - \lambda_1 s^{A*} - \lambda_F(s^{B*} - s^{A*})], \\ R^B(s^{A*}, s^{B*}) - R^B(s^{A*}, s^B) &= (p_1 - p_2)\lambda_F(s^{B*} - s^B) \geq 0. \end{aligned}$$

**Part II.3:**

$$R^B(s^{A*}, s^{B*}) \geq R^B(s^{A*}, s^B), \quad \text{for } s^{B*} \leq s^B \leq t.$$

In this case, firm  $A$  will be stocked out before the end of the horizon. First assume that firm  $B$  will switch the price before firm  $A$  is stocked out. Then,

$$\lambda_1 s^{A*} + \lambda_L(s^B - s^{A*}) + \lambda_2(\ell - s^B) = m^A. \quad (32)$$

Remember, we already have

$$\lambda_1 s^{A*} + \lambda_L(s^{B*} - s^{A*}) + \lambda_2(t - s^{B*}) = m^A \quad (33)$$

where  $\ell$  is the time that the firm  $A$  is stocked out.

At this point, we need to give the conditions under which firm  $B$  is not able to deplete all of its inventory by postponing its switching time. Otherwise, firm  $B$  can definitely increase its payoff so that  $(s^{A*}, s^{B*})$  is no longer an equilibrium. Let us find the minimum  $\lambda_{M2}$  that firm  $B$  can deplete its inventory before the end of the horizon. We should have a solution for the minimum  $\lambda_{M2}$  from the following equation:

$$\lambda_1 s^{A*} + \lambda_F(s^B - s^{A*}) + \lambda_2(\ell - s^B) + \lambda_{M2}(t - \ell) = m^B. \quad (34)$$

Using (32) and (34),

$$(\lambda_L - \lambda_F)(s^B - s^{A*}) + \lambda_{M2}(t - \ell) = m^A - m^B.$$

Noting that  $(\lambda_L - \lambda_F)(s^{A*} - s^{B*}) = m^A - m^B$ , we have

$$(\lambda_L - \lambda_F)(s^B - s^{B*}) - \lambda_{M2}(t - \ell) = 0.$$

Using (32) and (33), we have

$$(\lambda_L - \lambda_2)(s^B - s^{B*}) - \lambda_2(t - \ell) = 0. \quad (35)$$

Then we have,

$$\lambda_{M2} = \lambda_2 + \lambda_2 \frac{\lambda_2 - \lambda_F}{\lambda_L - \lambda_2}. \quad (36)$$

That is, firm  $B$  will be able to deplete all its inventory at time  $t$  only if  $\lambda_{M2}$  is given by equation (36). For

$$\lambda_{M2} \leq \lambda_2 \left( 1 + \frac{\lambda_2 - \lambda_F}{\lambda_L - \lambda_2} \right), \quad (37)$$

Firm  $B$  is not able to finish its inventory, if its switching time exceeds  $s^{B*}$ .

Now assuming that equation (37) is satisfied, we compare the payoff when  $s^B = s^{B*}$  and the payoff when  $s^{B*} \leq s^B \leq \ell$ .

$$\begin{aligned} R^B(s^{A*}, s^{B*}) &= p_1[\lambda_1 s^{A*} + \lambda_F(s^{B*} - s^{A*})] + p_2 \lambda_2(t - s^{B*}), \\ R^B(s^{A*}, s^B) &= p_1[\lambda_1 s^{A*} + \lambda_F(s^B - s^{A*})] \\ &\quad + p_2[\lambda_2(\ell - s^B) + \lambda_{M2}(t - \ell)], \\ R^A(s^{A*}, s^{B*}) - R^A(s^{A*}, s^B) &= (p_2 \lambda_2 - p_1 \lambda_F)(s^B - s^{B*}) \\ &\quad + p_2(\lambda_{M2} - \lambda_2)(t - \ell). \end{aligned}$$

Using equation (35),

$$R^A(s^{A*}, s^{B*}) - R^A(s^{A*}, s^B) = \left[ (p_2 \lambda_2 - p_1 \lambda_F) \frac{\lambda_2}{(\lambda_L - \lambda_2)} - p_2(\lambda_{M2} - \lambda_2) \right] (t - \ell)$$

which is positive only if

$$\lambda_{M2} \leq \lambda_2 \left[ 1 + \frac{p_2 \lambda_2 - p_1 \lambda_F}{p_2(\lambda_L - \lambda_2)} \right]. \quad (38)$$

**Part II.4:**

$$R^B(s^{A*}, s^{B*}) \geq R^B(s^{A*}, s^B), \quad \text{for } s^{B*} \leq s^B \leq t.$$

Now, assume that firm  $B$  will switch the price after firm  $A$ 's inventory is depleted. From Part II.3, we know that  $R^B(s^{A*}, s^{B*}) \geq R^A(s^{A*}, s^B)$  for  $s^B = \ell$ . Now assume  $s^B \geq \ell$ . Then,

$$R^A(s^{A*}, s^B) = p_1[\lambda_1 s^{A*} + \lambda_F(s^B - s^{A*}) + \lambda_{M1}(s^B - \ell)] + p_2\lambda_{M2}(t - s^B).$$

But,

$$\begin{aligned} R^A(s^{A*}, \ell) &= p_1[\lambda_1 s^{A*} + \lambda_F(\ell - s^{A*})] + p_2\lambda_{M2}(t - \ell), \\ R^A(s^{A*}, \ell) - R^A(s^{A*}, s^B) &= (p_2\lambda_{M2} - p_1\lambda_{M1})(s^B - \ell). \end{aligned}$$

For

$$p_2\lambda_{M2} - p_1\lambda_{M1} \geq 0, \quad (39)$$

$R^A(s^{A*}, \ell) - R^A(s^{A*}, s^B) \geq 0$  and  $R^A(s^{A*}, s^{B*}) \geq R^A(s^{A*}, s^B)$ . Combining the conditions (27), (29), (30), (31), (37), (38) and (39) all together,  $(s^{A*}, s^{B*})$  is a Nash equilibrium if

$$\lambda_{M2} \leq \lambda_2 \left( 1 + \min \left\{ \frac{p_2\lambda_L - p_1\lambda_1}{p_2(\lambda_1 - \lambda_F)}, \frac{p_2\lambda_2 - p_1\lambda_F}{p_2(\lambda_L - \lambda_2)} \right\} \right) \quad \text{and} \quad (40)$$

$$p_1\lambda_{M1} \leq p_2\lambda_{M2}. \quad (41)$$

For the linear demand model, one can show that  $\lambda_1 + \lambda_2 = \lambda_F + \lambda_L$  in a symmetric linear model, which also leads to,

$$\lambda_1 \leq \frac{\lambda_L + \lambda_F}{2} \leq \lambda_2.$$

Then,

$$\lambda_2 \left( 1 + \min \left\{ \frac{p_2\lambda_L - p_1\lambda_1}{p_2(\lambda_1 - \lambda_F)}, \frac{p_2\lambda_2 - p_1\lambda_F}{p_2(\lambda_L - \lambda_2)} \right\} \right) \geq 2\lambda_2. \quad (42)$$

But  $\lambda_{M2} = (1 + \theta)\lambda_2$  in a linear model, and  $\lambda_{M2} < 2\lambda_2$ , since  $\theta < 1$ , and condition (40) is always satisfied. Since  $p_1 + p_2 \geq 1/\beta$ , we also have  $p_1\lambda_{M1} \leq p_2\lambda_{M2}$ , which completes the proof.

Note that conditions (40) and (41) are the general conditions for the existence of the Nash equilibrium. Any demand model that satisfies these conditions will have a Nash equilibrium.

## B Proof of Theorem 2

To show the uniqueness of the equilibrium, we use Theorem 7.7. in Friedman (1977). We need to show that the response functions for both firms,  $r^A$  and  $r^B$  are contractions, i.e.,

$$\begin{aligned} |r^A(s_x^B) - r^A(s_y^B)| &< |s_x^B - s_y^B|, \text{ and} \\ |r^B(s_x^A) - r^B(s_y^A)| &< |s_x^A - s_y^A|. \end{aligned}$$

Note that each firm responds to its competitor's switching time in a way that it depletes its inventory right at time  $t$ , and response functions are piecewise linear functions. Below we focus on the response function for firm  $A$ . Five situations may be valid when firm  $A$  is responding to firm  $B$ . Note that only one of these cases will be valid, based on the parameters of the problem and the switching time of firm  $B$ .

**1.**  $A$ 's response is in such a way that it switches before  $B$  and  $B$  is not able to finish its inventory before  $t$ . The response function for  $A$  can be written as:

$$r^A(s^B) = \frac{\lambda_2 t - m^A}{\lambda_L - \lambda_1} + \frac{\lambda_L - \lambda_2}{\lambda_L - \lambda_1} s^B.$$

**2.**  $A$ 's response is in such a way that it switches before  $B$  and makes  $B$  run out of stock before  $t$ . The response function for  $A$  can be written as:

$$r^A(s^B) = \frac{\lambda_{M2} m^B + \lambda_2(m^A - m^B) - \lambda_2 \lambda_{M2} t}{\lambda_{M2}(\lambda_1 - \lambda_F) - \lambda_2(\lambda_L - \lambda_F)} + \frac{\lambda_{M2}(\lambda_2 - \lambda_F) - \lambda_2(\lambda_L - \lambda_F)}{\lambda_{M2}(\lambda_1 - \lambda_F) - \lambda_2(\lambda_L - \lambda_F)} s^B.$$

**3.**  $A$ 's response is in such a way that it switches after  $B$  and  $B$  runs out of stock after  $A$  switches and before  $t$ . The response function for  $A$  can be written as:

$$r^A(s^B) = \frac{\lambda_{M2} m^B + \lambda_2(m^A - m^B) - \lambda_2 \lambda_{M2} t}{\lambda_{M2}(\lambda_L - \lambda_2) - \lambda_2(\lambda_L - \lambda_F)} + \frac{\lambda_{M2}(\lambda_L - \lambda_1) - \lambda_2(\lambda_L - \lambda_F)}{\lambda_{M2}(\lambda_L - \lambda_2) - \lambda_2(\lambda_L - \lambda_F)} s^B.$$

**4.**  $A$ 's response is in such a way that it switches after  $B$  and  $B$  runs out of stock before  $A$  switches and before  $t$ . The response function for  $A$  can be written as:

$$r^A(s^B) = \frac{\lambda_L m^A - (\lambda_{M1} - \lambda_F) m^B + \lambda_L \lambda_{M2} t}{\lambda_L(\lambda_{M2} - \lambda_{M1})} + \frac{\lambda_1(\lambda_L - \lambda_F) - \lambda_{M1}(\lambda_L - \lambda_1)}{\lambda_L(\lambda_{M2} - \lambda_{M1})} s^B.$$

**5.**  $A$ 's response is in such a way that it switches after  $B$  and  $B$  is not able to finish its inventory before  $t$ . The response function for  $A$  can be written as:

$$r^A(s^B) = \frac{\lambda_2 t - m^A}{\lambda_2 - \lambda_F} + \frac{\lambda_1 - \lambda_F}{\lambda_2 - \lambda_F} s^B.$$

To prove that  $r^A$  is a contraction, we only need to show that each possible piece of  $r^A$  has a slope whose absolute value is less than 1. The absolute value of each slope is less than 1, if the following conditions hold:

$$\lambda_{M2} < \lambda_2 \left( 1 + \min \left\{ \frac{2\lambda_L - (\lambda_1 + \lambda_2)}{(\lambda_1 + \lambda_2) - 2\lambda_F}, \frac{(\lambda_1 + \lambda_2) - 2\lambda_F}{2\lambda_L - (\lambda_1 + \lambda_2)} \right\} \right), \quad (43)$$

$$\left| \frac{\lambda_1(\lambda_L - \lambda_F) - \lambda_{M1}(\lambda_L - \lambda_1)}{\lambda_L(\lambda_{M2} - \lambda_{M1})} \right| < 1. \quad (44)$$

Note that not all pieces will be feasible for a firm. Similar functions and conditions are required for firm  $B$  and equations (43) and (44) are the sufficient conditions for  $(s^{A*}, s^{B*})$  to be a unique equilibrium.

For the linear demand model,  $\lambda_{M2} = (1 + \theta)\lambda_2$  and  $\lambda_1 + \lambda_2 = \lambda_L + \lambda_F$  and (43) is reduced to  $\theta < 1$  which is always true. We also have,  $\lambda_L - \lambda_F = \frac{1+\theta}{1-\theta}(\lambda_2 - \lambda_1)$  and  $\lambda_L - \lambda_1 = \frac{1}{1-\theta}(\lambda_2 - \lambda_1)$  which reduces the left hand side of equation (44) to 0, which completes the proof.

Note that conditions (43) and (44) are the general conditions for the uniqueness for any demand model. The sufficiency bounds of  $\lambda_{M1}$  and  $\lambda_{M2}$  for uniqueness may be tighter than the conditions (40) and (41) for equilibrium given in Theorem 1.