

Component Goods and Innovation

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Abstract

We consider the research and development (R&D) decisions of a durable good monopolist that can only engage in partial physical obsolescence—the obsolescence of component parts. We show that if the durable good producer also monopolizes the component good market then he chooses the socially efficient level of R&D. In contrast, if the component good market is competitive then the monopolist gets practically no benefit from being able to make components obsolete. In general if the component good market gets more competitive in the future then the amount of current R&D decreases.

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1 INTRODUCTION

Durability choice and the related issue of planned obsolescence are important concerns of analysis in durable good monopoly theory. In this literature it has always been assumed that the monopolist is able to limit the durability of the whole of the product.¹ However for most durable goods—such as cellphones, printers, automobiles, computers, and stereo systems—it is not the whole unit that becomes completely obsolete, but a component of the unit. It is the battery not the cellphone, the toner cartridge not the photocopier, the car parts not the car that dies first. Casual observations show that replacing a component may be as expensive as buying a new product. Consider a Xerox Phaser 3117 printer. The price of this model is \$104, while the service cost by Xerox is \$35 plus the price of the component that needs to be replaced. A toner cartridge costs \$90 and thus if one needs both a minor repair and a toner cartridge it is not worth fixing the printer.² While the pricing behavior in component good markets has gained some attention, the impact of these markets on the firm’s research and development (R&D) has not been analyzed.

In this paper we study the R&D incentives of a durable good monopolist that can only engage in partial physical obsolescence—the obsolescence of component parts. We use the model in Fishman and Rob (2000) and adopt it to our setting. Fishman and Rob (2000) investigate a durable good monopolist’s R&D incentives under the assumption that innovations are recurrent and knowledge builds up cumulatively. They find that the monopolist innovates less frequently and invests less than the efficient level. The reason for this is that when a new model appears, the older model is still functional. As a result the new model competes with the old, and consumers are willing to pay only for the incremental flow of services provided by the new model—until it is replaced by a further upgrade. However, the introduction of a new model increments consumers’ utility in perpetuity since the new model forms the technological base for the subsequent models. Fishman and Rob (2000) show that if the monopolist is able to make the old model obsolete when the new model is introduced then the monopolist gets the social value of each model and is induced to innovate at the socially optimal pace.

¹Planned obsolescence is defined as the production of goods with uneconomically short useful lives (Bulow (1986)).

²The quotes for sales prices are from Office Superstore 1, the quote for the service cost is from Aroks, a Xerox approved service center, both located in Ankara Turkey.

In practice it might be quite difficult if not impossible to make the whole product obsolete at once. If a component breaks down then the consumer has to decide whether to replace the component or buy a new product. Therefore, the competition for the new generation is essentially provided by the old generation with a replaced component. Hence in addition to affecting pricing decisions, the competitive environment of the component good market has a critical impact on the R&D incentives of the firm. In our paper, the monopolist introduces the new model when the component of the previous model becomes obsolete. Now the monopolist is able to charge consumers for both the incremental utility and the replacement cost of the component. When there is perfect competition in the component market, our analysis is the same as Fishman and Rob's analysis without obsolescence; there is too little innovation. In general when the component good market gets more competitive, the amount of R&D decreases. When the durable good producer also monopolizes his component good market he chooses the socially optimal level of research and development.

The insights of this study also contribute to the aftermarket literature. The term "aftermarkets" refers to markets for complementary goods and services such as maintenance and replacement parts that may be needed after a consumer has purchased a durable good. In this context, components market can also be considered an aftermarket. A series of court cases involving firms such as Kodak, Data General, Unisys and Xerox concern the issue of aftermarket monopolization. There are several theories that try to explain why a durable good producer tries to reduce competition in aftermarkets. Chen and Ross (1993) observe that many of the anti-trust cases have involved allegations that manufacturers of durable goods have refused to supply parts to independent service organizations, apparently to monopolize the market for repairs of their products. Hence they consider a model in which the durable good monopolist monopolized the market for repairs through its refusal to supply propriety parts to independent service organizations. Chen and Ross (1998); Borenstein, Mackie-Mason and Netz (2000) study such refusals in a competitive market and its connected aftermarket. One important question that the literature tries to answer is whether the monopolization of the aftermarket by the original equipment manufacturer causes an efficiency loss.

Chen and Ross (1993) show that these refusals could be a part a monopolist's program of third degree price discrimination in which high intensity-high value consumers pay a substantial premium for the product via the expensive repairs that they disproportionately consume. They conclude that since the third degree

price discrimination could be welfare increasing or reducing the effect of the monopolization of this aftermarket on welfare is ambiguous. Chen and Ross (1998) argue that refusals help support higher prices for high value users but at the same time permit the recovery of higher costs incurred during an initial warranty period. Since full prices reflect full marginal costs at equilibrium, the refusals permit the attainment of a first-best outcome. Accordingly, an attempt by anti-trust authorities to force supply would be welfare reducing. Borenstein, Mackie-Mason and Netz (2000) on the other hand, show that price in the services and parts market will exceed marginal cost despite the competition in the primary good market.

Our paper contributes to this literature by showing that the competitive environment of aftermarkets will have a significant impact on the R&D decisions of the primary good producer. We would like to argue that while our discussion focuses on durable goods monopolists, this does not mean that the insights from our analysis only apply to such settings. Many durable goods producers have some market power and hence monopoly analysis should provide useful insights.

In the next section, we present relevant literature on planned obsolescence. In section 3 we describe our model, and then in the next two sections we analyze the two boundary cases for competition in the component good market. In section 4 we assume the monopolist also has a monopoly in the component good market, and in section 5 we assume the component good market is perfectly competitive. In section 6 we look at the case where the component good market is imperfectly competitive. Section 7 concludes.

2 Related Literature

Fishman and Rob (2000) is related to the Coase Conjecture (1972) in the sense that planned obsolescence is a business strategy that helps a durable good monopolist to maintain its market power. Coase (1972) considered the dynamic pricing problem of a monopoly selling a durable good (of fixed quality) to consumers with different valuations. Coase argued that if the monopoly is unable to commit to future prices (or equivalently, commit not to sell any more units), the price must eventually fall as the market clears of high valuation buyers in the second period. In that setting, planned obsolescence restores the monopoly's ability to charge monopoly prices. In the setting of Fishman and Rob (2000), product durability limits market power by increasing the value of the consumers' outside option and thereby reducing their willingness to pay for new models.

Hence limiting product durability reduces the value of consumers' outside options. However, we should note that while the ability to precommit to future sales completely solves the Coasian monopoly's problem, a similar ability to precommit to future introduction dates does not accomplish the same purpose in the context of recurring innovations. It is also important to note that in the Coasian setting, market efficiency is improved as prices fall, but in Fishman and Rob a monopolist that can not charge for the social value of an innovation innovates less than the socially efficient amount.

Waldman (1993, 1996), Choi (1994), Kumar (2002) have also considered that the introduction of a new product can lower the value of used units. Waldman (1996) demonstrates that a similar result to that in Coase (1972) and Bulow (1982) holds within the context of the monopolist's R&D expenditures. Since the monopolist does not internalize in the second period how its behavior affects the value of units sold previously, the monopolist's incentive to invest in R&D that makes past production "technologically obsolete" is too high. Hence in that paper the term planned obsolescence is used to mean that the monopolist has an incentive to engage in R&D decisions and new products introductions and thereby make the past production technologically obsolete. Waldman finds that although time inconsistency causes overinvestment in R&D from the standpoint of the monopolist's own profitability, from the standpoint of social welfare the time inconsistency problem is in fact beneficial. He finds that in the case where the monopolist can commit to a future value for R&D, the firm is unable to capture all the societal benefits from the improved quality of its output. As a result the private incentive to invest in R&D is less than the incentive that is social welfare maximizing.

Any action that affects the value of the used units previously sold—such as an R&D decision that lowers the value of used units—can be subject to time inconsistency. Waldman (2003) argues that the problem of time inconsistency is potentially more important for choices other than output. Contractual provisions allow firm to commit at least to some extent to future outputs and prices. In contrast the size and nature of R&D investments would seem much more difficult to specify in contracts in an enforceable manner.

3 Model

There is a continuum of identical infinitely lived consumers of measure one, and an infinitely durable product. The utility of a representative consumer from

the product depends on its quality, denoted q . A product of quality q delivers q units of utility per unit of time. Consumers have unit demands: They can use either zero or one unit but have no use for a second unit. Time is continuous, and the instantaneous interest rate $r > 0$ is constant and the same for consumers and producers.

New and improved models are introduced over time. The introduction of a new model is preceded by an R&D stage, called a *gestation period*. The length of this period is t , and the flow R&D expenditures during this period are x , x is constant throughout the gestation period. The quality improvement is determined by an *innovation function* $g(x, t)$. If an old model is of quality q and x is spent over t units of time then the new model has a quality of $q + g(x, t)$. There are recurrent introductions, and the introduction of one model triggers the gestation period of the next. Quality improvements are cumulative, so if the R&D inputs are $\{x_1, t_1\}$ and $\{x_2, t_2\}$ then the quality of the second-generation model is $q + g(x_1, t_1) + g(x_2, t_2)$.

In addition to R&D expenditure, a fixed cost of F has to be paid each time a new product is introduced and there is a constant marginal cost of c . These are both paid at the time the product is sold, F is an implementation cost and c is the cost of production. We make the following assumptions about the primitives of the model.

Assumption 1 g is strictly concave, bounded, increasing in x and t , twice continuously differentiable, $g(x, 0) = g(0, t) = 0$, and there exists an $\{x_p, t_p\}$ such that $r(F + c) < (1 - e^{-rt_p}) [g(x_p, t_p) - e^{rt_p} x_p]$.

All but the last assumption are made merely to simplify analysis. The last assumption guarantees that all monopolists will invest in research and development. We follow Fishman and Rob (2000) in assuming that x and t are substitutes for the optimizing firm. Our assumption is weaker than theirs because they only analyze the case where x and t are constant in all stages.

Assumption 2 x and t are substitutes in optimization, or

$$(1 - e^{-rt}) g_{xt} \leq r g_x . \tag{1}$$

Up to this point our model and assumptions are exactly the same as Fishman and Rob. Since we are doing comparative static analysis we also want to make sure that the objective function is strictly concave.

Assumption 3 The objective function is strictly concave, or

$$\left((1 - e^{-rt})g_{xt} - rg_x\right)^2 \leq \frac{1}{4} \frac{1 - e^{-rt}}{e^{-rt}} g_{xx}g_{tt}. \quad (2)$$

These conditions can both be met for a family of Cobb-Douglas innovation functions. Specifically condition 1 is met for all Cobb-Douglas innovation functions, while condition 2 is met if the coefficient on x is small enough relative to the coefficient on t . To be exact if $g(x, t) = Ax^\chi t^\tau$ we need $\chi \leq \frac{\tau}{16-15\tau}$.

The critical difference between Fishman and Rob (2000) and our analysis is that we assume there is a finitely durable component good which is a perfect complement to the primary good. As the new model of the primary product is introduced, the component becomes obsolete. We assume the component good is produced at a cost of αc , where $\alpha \in (0, 1)$. Thus, the production cost of the component is just a constant fraction of the marginal cost of the primary product.

The primary focus of our analysis will be the impact of competition in the component good market. If we were to do a structural analysis of this market then we might overlook an important alternative model, thus we will summarize the competitive nature of this market with a reduced form inverse demand curve, $p^c(g, q, \alpha c)$. We include q in this price since component suppliers might take advantage of the high value of the primary good to charge more for the component. We include g for a similar reason—if the new model is going to be astronomically expensive then the component price could rise. We include αc since this is the cost of producing the component good. Our analysis would not be affected by removing any of these arguments from this inverse demand curve, in fact it would be simpler. For simplicity we make most of the same assumptions on p^c as we do on g .

Assumption 4 $p^c(g, q, \alpha c)$ is strictly concave, increasing in all arguments, and twice continuously differentiable.

Notice that for simplicity we assume that when the component is replaced, the good is exactly the same quality as it was before the component broke down. This assumption is without loss of generality because the function $p^c(g, q, \alpha c)$ can also address the case where the good with a replaced component is of lower quality.

Throughout our analysis the monopolist sells its products for one price to all buyers, and there is no secondhand market for the primary product. We

will also assume that $\{x, t\}$ are based only on the state of the system or q , furthermore if possible the monopolist prefers to choose the same $\{x, t\}$ for each model.

4 Monopolistic Component Good Market

Clearly the best case for the monopolist is when they also have a monopoly in the component good market. In this case they can also achieve the globally optimal solution—the solution that maximizes social welfare.

First we will show how the monopolist chooses the price of the component and the price of the primary good. Let p be the price of the new model and t_e be the length of time that the consumer expects the new model to last. Then consumers with a quality q good in their possession will buy a quality $q+g$ good at a price of p if $(q+g)\frac{1-e^{-rt_e}}{r} - p \geq q\frac{1-e^{-rt_e}}{r} - p^c$ and $(q+g)\frac{1-e^{-rt_e}}{r} - p \geq 0$. Thus

$$p \leq \min \left\{ (q+g)\frac{1-e^{-rt_e}}{r}, g\frac{1-e^{-rt_e}}{r} + p^c \right\} \quad (3)$$

and a monopolist will choose $p^c = \frac{1-e^{-rt_e}}{r}q$ so that $p = (q+g)\frac{1-e^{-rt_e}}{r}$. The Bellman equation is

$$\Pi^m(q) = \max_{x,t} \left\{ -x\frac{1-e^{-rt}}{r} + e^{-rt}\frac{1-e^{-rt_e}}{r}(q+g(x,t)) - e^{-rt}(F+c) + e^{-rt}\Pi^m(q+g(x,t)) \right\}. \quad (4)$$

(We use a m superscript to indicate that the monopolist also has a monopoly in the component good market.) If we let $\hat{\Pi}(q) = \Pi^m(q) + q\frac{1-e^{-rt_e}}{r}$ then the profit maximization problem can be re-written as

$$\hat{\Pi}(q) = \max_{x,t} \left\{ q\frac{1-e^{-rt_e}}{r} - x\frac{1-e^{-rt}}{r} - e^{-rt}(F+c) + e^{-rt}\hat{\Pi}(q+g) \right\}. \quad (5)$$

We can make this transformation without loss of generality since q is a state variable and t_e is the expectations of the consumers. Let us compare this with the social welfare Bellman equation

$$W(q) = \max_{x,t} \left\{ (q-x)\frac{1-e^{-rt}}{r} - e^{-rt}(F+c) + e^{-rt}W(q+g(x,t)) \right\} \quad (6)$$

Since $W(q)$ and $\hat{\Pi}(q)$ are the same function they have the same optimal value.

Proposition 1 *The monopolist who has a monopoly in the component good market chooses the socially optimal level of research and development.*

Proof. In equilibrium $t_e = t$ thus the objective functions in 5 is the same as the objective function in 6 in equilibrium. Thus the maximum for welfare also maximizes profits. ■

5 Competitive Component Good Market

This is the worst case for the monopolist. In this case clearly $p^c = \alpha c$ as implied by a perfectly competitive component good market. Thus $p = g \frac{1-e^{-rt_e}}{r} + \alpha c$ ³ and the objective function is

$$\Pi^c(q) = \max_{x,t} \left\{ -x \frac{1-e^{-rt}}{r} + e^{-rt} g(x,t) \frac{1-e^{-rt_e}}{r} - e^{-rt} (F + (1-\alpha)c) + e^{-rt} \Pi^c(q + g(x,t)) \right\}. \quad (7)$$

(We use a c superscript to indicate that the component good market is competitive.) Notice that the component increases the price by αc without changing the quantity sold. The effect of this price increase on profits is equivalent to the effect of a decrease in production costs. In other words, we would get the same effect on profits had there been a reduction in production costs from c to $(1-\alpha)c$. Note that in equation 7, q only appears as a state variable, thus $\Pi^c(q)$ is actually independent of q . Since the monopolist prefers to always choose the same $\{x, t\}$ when he can, we can write the objective function as

$$\Pi^c = \max_{x,t} \left(-\frac{x}{r} + \frac{e^{-rt}}{1-e^{-rt}} g(x,t) \frac{1-e^{-rt_e}}{r} - \frac{e^{-rt}}{1-e^{-rt}} (F + (1-\alpha)c) \right). \quad (8)$$

Compare the objective function in equation 7 to the worst case in Fishman and Rob (2000), when the monopolist can not control the obsolescence of the good,

$$\Pi(q) = \max_{x,t} \left\{ -x \frac{1-e^{-rt}}{r} + e^{-rt} \frac{1-e^{-rt_e}}{r} g(x,t) - e^{-rt} (F + c) + e^{-rt} \Pi(q + g(x,t)) \right\} \quad (9)$$

one can immediately see that the only difference between the two equations is that production costs are higher in the latter equation. They are c instead of $(1-\alpha)c$. This observation immediately gives us the following proposition.

Proposition 2 *When the component good market is competitive the equilibrium amount of innovation is the same as the amount without planned obsolescence when the production costs are lowered by αc .*

³We assume that quality is high enough that $q \frac{1-e^{-rt_e}}{r} \geq \alpha c$. If this is not true no one will consider replacing the component and the component good market is irrelevant.

The lowering of costs has a trivial impact on the amount of innovation. When the monopolist also has a monopoly in the component good market the main motive for innovation is that it will increase the price charged for every future model. In contrast in this case the only benefit is to increase the price charged for the current model since the current level of quality— q —does not affect the price any model. Thus competition in the component market is essentially the same as the monopolist not being able to control the obsolescence of the good. It is unimportant that the monopolist can use planned obsolescence, there is a trivial benefit from using it.

6 Imperfectly Competitive Component Good Market

In this case the condition for the price of the new model is the same as in section 4,

$$p \leq \min \left\{ (g + q) \frac{1 - e^{-rt_e}}{r}, g \frac{1 - e^{-rt_e}}{r} + p^c \right\} \quad (10)$$

and without loss of generality we can assume $p^c \leq q \frac{1 - e^{-rt_e}}{r}$, thus the price the monopolist will charge will be $p = g + p^c$. The Bellman equation is

$$\Pi(q) = \max_{x,t} \left\{ -x \frac{1 - e^{-rt}}{r} + e^{-rt} p^c(g, q, \alpha c) + e^{-rt} \frac{1 - e^{-rt_e}}{r} g(x, t) - e^{-rt} (F + c) + e^{-rt} \Pi(q + g(x, t)) \right\}. \quad (11)$$

Now we can get to the key proposition of our paper, showing that if the component good market becomes more competitive then the monopolist spends less on R&D, resulting in socially inefficient innovation.

Proposition 3 *If the competitiveness in the component good market increases at any point in the future (p^c decreases) then the monopolist decreases the investment in R&D and increases the gestation periods. Thus a more competitive component good market reduces the amount of innovation.*

Proof. Let $p^c(k)$ be the price of the component good of the model k stages from now. Then by the envelope theorem it is immediate that $\Pi_{p^c(k)} = e^{-rt} e^{-r \sum_{j=1}^k t(j)} > 0$ where we denote the value of t chosen j stages from now as $t(j)$. Thus $\Pi_{xp^c(k)} = 0$ and $\Pi_{tp^c(k)} = -r \Pi_{p^c(k)}$. Using Cramer's rule we find

$$\begin{aligned} \frac{\partial x}{\partial p^c(k)} &= \frac{-r \Pi_{p^c(k)} \Pi_{xt}}{\Pi_{xx} \Pi_{tt} - \Pi_{xt}^2} \\ \frac{\partial t}{\partial p^c(k)} &= \frac{r \Pi_{p^c(k)} \Pi_{xx}}{\Pi_{xx} \Pi_{tt} - \Pi_{xt}^2}. \end{aligned}$$

In Lemma ?? we show that $\Pi_{xx} < 0$, $\Pi_{xx}\Pi_{tt} - \Pi_{xt}^2 > 0$ thus $\frac{\partial t}{\partial p^c(k)} < 0$. We are done when we show that $\frac{\partial x}{\partial p^c(k)} > 0$ or that $\Pi_{xt} < 0$.

$$\Pi_{xt} = -1 + \frac{1}{g_x} \frac{1 - e^{-rt}}{r} g_{xt} + e^{-rt} (p_{gg}^c + \Pi_{qq}(q + g)) g_x g_t < 0$$

Now $p_{gg}^c < 0$ and $\Pi_{qq}(q) = e^{-rt} p_{qq}^c + \Pi_{qq}(q + g) < 0$ thus it is sufficient if $(1 - e^{-rt}) g_{xt} \leq r g_x$ which we have assumed. ■

7 Conclusion

Early work in durable goods theory has modelled planned obsolescence as reduction in the durability of the whole of the product. However, it is typically a component that becomes physically obsolete first rather than the whole unit.

In this paper, we analyze the R&D investments and the frequency of product innovations of a durable good monopoly under the assumption that the monopolist determines the durability of the component. We study the effects of the market structure in the component market on the R&D decisions of the firm. The main assumptions of the model are that innovations are recurrent, the knowledge builds up cumulatively, consumers are homogenous and the monopolist sells rather than rent its products. We show that under these circumstances if the monopolist has complete market power in the component market, he innovates at the socially optimal pace. If the competitiveness in the component good market increases at any point in the future then the monopolist decreases the investment in R&D and increases the gestation periods. Thus a more competitive component good market reduces the amount of innovation in the primary good market. Simply put new models are introduced less frequently and the innovation in each new model is less than it should be; resulting in a loss to society.

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8 Appendix

Lemma 4 *The Monopolists objective function is strictly concave at the optimum.*

Proof. *The first order conditions are*

$$\begin{aligned}\Pi_x &= -\frac{1 - e^{-rt}}{r} + e^{-rt} \left(p_g^c + \frac{1 - e^{-rt_e}}{r} + \Pi_q(q + g) \right) g_x \\ \Pi_t &= -r \left(\Pi(q) + \frac{x}{r} \right) + e^{-rt} \left(p_g^c + \frac{1 - e^{-rt_e}}{r} + \Pi_q(q + g) \right) g_t\end{aligned}$$

from these one can derive that the second derivatives at the optimum are:

$$\begin{aligned}\Pi_{xx} &= \frac{1}{rg_x} g_{xx} + e^{-rt} g_x^2 (p_{gg}^c + \Pi_{qq}(q+g)) \\ \Pi_{tt} &= (g_{tt} - rg_t) \left(\Pi(q) + \frac{x}{r} \right) \frac{r}{g_t} + e^{-rt} g_t^2 (p_{gg}^c + \Pi_{qq}(q+g))\end{aligned}$$

these are both strictly negative since $p_{gg}^c < 0$ and $\Pi_{qq}(q) = e^{-rt} p_{qq}^c + e^{-rt} \Pi_{qq}(q+g) < 0$. Now before we proceed it is desirable to get rid of the term $\Pi(q) + \frac{x}{r}$. We can do this since $\Pi_{xt} = \Pi_{tx}$.

$$\begin{aligned}\Pi_{xt} &= -1 + \frac{1 - e^{-rt}}{r} \frac{g_{tx}}{g_x} + e^{-rt} g_x g_t (p_{gg}^c + \Pi_{qq}(q+g)) \\ \Pi_{tx} &= -1 + \frac{g_{tx}}{g_t} e^{-rt} r \left(\Pi(q) + \frac{x}{r} \right) + e^{-rt} g_t g_x (p_{gg}^c + \Pi_{qq}(q+g(x,t)))\end{aligned}$$

Thus $\Pi(q) + \frac{x}{r} = \frac{g_t}{g_x} \frac{1 - e^{-rt}}{r^2 e^{-rt}}$ and

$$\begin{aligned}\Pi_{xx} \Pi_{tt} - \Pi_{xt}^2 &= \frac{1}{r^2 g_x^2} g_{xx} (g_{tt} - rg_t) \frac{1 - e^{-rt}}{e^{-rt}} - \left(-1 + \frac{1 - e^{-rt}}{r} \frac{g_{xt}}{g_x} \right)^2 \\ &\quad + \left(\frac{1}{rg_x} g_{xx} e^{-rt} g_t^2 + (g_{tt} - rg_t) \frac{1 - e^{-rt}}{r} g_x - 2 \left(-rg_x + \frac{1 - e^{-rt}}{r} g_{xt} \right) \frac{e^{-rt}}{r} g_t \right) \\ &\quad * (p_{gg}^c + \Pi_{qq}(q+g)) .\end{aligned}$$

In order for this to be strictly positive we need:

$$\begin{aligned}\frac{1}{r^2 g_x^2} g_{xx} (g_{tt} - rg_t) \frac{1 - e^{-rt}}{e^{-rt}} &> \left(-1 + \frac{1 - e^{-rt}}{r} \frac{g_{xt}}{g_x} \right)^2 \\ g_{xx} (g_{tt} - rg_t) \frac{1 - e^{-rt}}{e^{-rt}} &> ((1 - e^{-rt}) g_{xt} - rg_x)^2 \\ \frac{1}{2} g_{xx} \frac{g_t}{g_x} + \frac{1}{2} (g_{tt} - rg_t) \frac{1 - e^{-rt}}{e^{-rt}} \frac{g_x}{g_t} &< (1 - e^{-rt}) g_{xt} - rg_x\end{aligned}$$

Since Now since $\frac{1 - e^{-rt}}{e^{-rt}} \frac{1}{4} g_{xx} g_{tt} \geq ((1 - e^{-rt}) g_{xt} - rg_x)^2$ the first inequality is met strictly. In the second expression since both sides are negative this is equivalent to asserting:

$$\left(\frac{1}{2} \frac{g_t}{g_x} g_{xx} + \frac{1}{2} (g_{tt} - rg_t) \frac{1 - e^{-rt}}{e^{-rt}} \frac{g_x}{g_t} \right)^2 > (-rg_x + (1 - e^{-rt}) g_{xt})^2$$

In the quadratic on the right hand side every term is positive and one is: $\frac{1}{2} (g_{tt} - rg_t) \frac{1 - e^{-rt}}{e^{-rt}} \frac{g_x}{g_t} \frac{1}{2} \frac{g_t}{g_x} g_{xx} > \frac{1 - e^{-rt}}{e^{-rt}} \frac{1}{4} g_{xx} g_{tt} \geq ((1 - e^{-rt}) g_{xt} - rg_x)^2$ thus the second inequality is met strictly. ■